

K-area, Hofer metric and geometry of conjugacy classes in Lie groups

(PRELIMINARY VERSION)

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Abstract

Given a closed symplectic manifold (M, ω) we introduce a certain quantity associated to a tuple of conjugacy classes in the universal cover of the group $Ham(M, \omega)$ by means of the Hofer metric on $Ham(M, \omega)$. We use pseudo-holomorphic curves involved in the definition of the multiplicative structure on the Floer cohomology of a symplectic manifold (M, ω) to estimate this quantity in terms of actions of some periodic orbits of related Hamiltonian flows. As a corollary we get a new way to obtain Agnihotri-Belkale-Woodward inequalities for eigenvalues of products of unitary matrices. As another corollary we get a new proof of the geodesic property (with respect to the Hofer metric) of Hamiltonian flows generated by certain autonomous Hamiltonians. Our main technical tool is K-area defined for Hamiltonian fibrations over a surface with boundary in the spirit of L. Polterovich's work on Hamiltonian fibrations over S^2 .

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1 Introduction and overview of the main results

1.1 Definitions and previously known results concerning Δ_l^G

With a connected Lie group G one can associate an important object Δ_l^G defined below.

Definition 1.1.1 Define Δ_l^G as the set of all l -tuples of conjugacy classes $(\mathcal{C}_1, \dots, \mathcal{C}_l)$ in G such that $\varphi_1 \cdot \dots \cdot \varphi_l = Id$ for some $\varphi_i \in \mathcal{C}_i$, $i = 1, \dots, l$.

If $G = SU(n)$ (see e.g. [1]) the structure of Δ_l^G contains the answer to the following question: given $l-1$ matrices $A_1, \dots, A_{l-1} \in SU(n)$, what can be said about the eigenvalues of the product $A_1 \cdot \dots \cdot A_{l-1}$ in terms of the eigenvalues of the factors? In the case $G = GL(n, \mathbb{C})$ the structure of Δ_l^G is the subject of the so called *Deligne-Simpson problem* and is important in studying possible monodromies of a multi-valued solution of a regular Fuchsian system of differential equations on the Riemannian sphere – see e.g. [48].

When the Lie group G is compact, connected and finite dimensional it is possible in some cases to get a nice description of Δ_l^G in geometric terms. Let \mathfrak{g} be the Lie algebra of G and let \mathfrak{t} be the Lie algebra of a maximal torus in G . Choose a fundamental domain $\mathfrak{U} \subset \mathfrak{t}$ for the adjoint action of G on \mathfrak{g} so that points in \mathfrak{U} parameterize all the conjugacy classes in G : to each conjugacy class \mathcal{C} in G one associates the unique point $x \in \mathfrak{U}$ such that $\exp(x) \in \mathcal{C}$. Such a domain \mathfrak{U} is not unique. In the case when G is semi-simple one can choose \mathfrak{U} as:

$$\mathfrak{U} = \{\xi \in \mathfrak{t}_+ \mid \alpha_0(\xi) \leq 1\},$$

where \mathfrak{t} is the Lie algebra of a maximal torus in G , \mathfrak{t}_+ is a positive Weyl chamber and α_0 is the highest root. Thus Δ_l^G can be viewed as a subset $\Delta_l^G \subset \mathfrak{U}^l \subset \mathfrak{t}^l$. A theorem of Meinrenken and Woodward [26] states that if the finite dimensional, compact, connected Lie group G is also simply connected, then $\Delta_l^G \subset \mathfrak{t}^l$ is a convex polytope of maximal dimension. (This is essentially due to the convexity of the image of a certain moment map).

If $G = SU(n)$ the Agnihotri-Belkale-Woodward theorem (see [1],[4]) gives a complete description of the convex polytope Δ_l^G : the inequalities defining Δ_l^G (we will call them *ABW inequalities*) are in one-to-one correspondence with non-zero l -point spherical Gromov-Witten invariants of all

the complex Grassmannians $Gr(r, n)$, $1 \leq r \leq n - 1$. In other words, the Agnihotri-Belkale-Woodward theorem provides a complete list of inequalities describing possible eigenvalues of a product of unitary matrices in terms of the eigenvalues of the factors.

1.2 Introducing a pseudo-metric on the group

In this paper we suggest a new way to study Δ_l^G . Namely we equip the tangent bundle of G with a *bi-invariant* norm, i.e. each tangent space is equipped with a norm which varies smoothly with the base point. We will call such a norm *Finsler*. (Sometimes, given certain additional assumptions on smoothness and convexity of the norm outside of the zero section, such a structure is called *an absolutely homogeneous Finsler structure* on G [3]). One can measure lengths of paths in the group with respect to a Finsler norm on the tangent bundle and define the distance between two points in the group as the infimum of lengths of paths connecting them. This defines a *Finsler bi-invariant pseudo-metric* ρ on G , i.e. it is a bi-invariant symmetric function on $G \times G$ which satisfies the triangular inequality but may vanish not only on the diagonal but also outside of it. If $\rho(x, y) \neq 0$ as long as $x \neq y$ then the pseudo-metric ρ is a genuine *Finsler metric* on G . If G is finite-dimensional then such a Finsler norm on T_*G always defines a genuine Finsler metric on the group. In the infinite-dimensional case this may not be the case.

An important example of a group with a bi-invariant Finsler norm is provided a certain C^0 -norm on the tangent bundle of the infinite-dimensional Lie group $Ham(M, \omega)$ defining the famous Hofer metric on the group – see the definitions in Section 1.3. (More precisely, we will mostly consider the universal cover $\widetilde{Ham}(M, \omega)$ of $Ham(M, \omega)$ equipped with the pullback of the norm on the tangent bundle of $Ham(M, \omega)$ under the covering map).

Definition 1.2.1 Given a bi-invariant Finsler pseudo-metric ρ on G and a tuple $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_l)$ of conjugacy classes in G define

$$\Upsilon_l(\mathcal{C}) = \inf_{\varphi_i \in \mathcal{C}_i} \rho(Id, \varphi_1 \cdot \dots \cdot \varphi_l).$$

The quantity Υ_l has the following elementary properties.

- 1) If $\Upsilon_l(\mathcal{C}) = 0$ then $\mathcal{C} \in \Delta_l^G$ and if G is finite-dimensional then $\Delta_l^G = \{\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_l) \mid \Upsilon_l(\mathcal{C}) = 0\}$.

- 2) $\Upsilon_l(\mathcal{C}_1, \dots, \mathcal{C}_l) = \inf_{\mathcal{C}_{l+1}} \rho(Id, \mathcal{C}_{l+1})$, where the infimum is taken over all \mathcal{C}_{l+1} such that $(\mathcal{C}_1, \dots, \mathcal{C}_l, \mathcal{C}_{l+1}) \in \Delta_{l+1}^G$.
- 3) Υ_l does not depend on the ordering of classes in a tuple – it is actually a function on *sets* of conjugacy classes.
- 4) $\Upsilon_2(\mathcal{C}_1, \mathcal{C}_2) = \rho(\mathcal{C}_1^{-1}, \mathcal{C}_2)$, where the right-hand side denotes the distance between the two closed sets in the pseudo-metric space.
- 5) $\Upsilon_l(\mathcal{C}_1, \dots, \mathcal{C}_l) = \Upsilon_l(\mathcal{C}_1^{-1}, \dots, \mathcal{C}_l^{-1})$.
- 6) Triangular inequality: $\Upsilon_l(\mathcal{C}_1, \dots, \mathcal{C}_l) \geq \rho(Id, \mathcal{C}_l) - \rho(Id, \mathcal{C}_{l-1}) - \dots - \rho(Id, \mathcal{C}_1)$.

Observe that the property 1) is true no matter which Finsler pseudo-metric ρ on G we take. Thus a good choice of the bi-invariant Finsler pseudo-metric may help us in understanding the structure of Δ_l^G . Such a good choice will come from the pullback of the Finsler norm on the tangent bundle of $Ham(M, \omega)$ defining the Hofer metric under a homomorphism $G \rightarrow Ham(M, \omega)$ representing a Hamiltonian action of G on a closed symplectic manifold (M, ω) . As an example consider the case of $G = SU(n)$ acting on $(\mathbf{CP}^{n-1}, \omega)$, where ω is the standard (Fubini-Study) symplectic form. The action is Hamiltonian and thus it defines a homomorphism $SU(n) \rightarrow Ham(\mathbf{CP}^{n-1}, \omega)$ with a the kernel \mathbf{Z}_n . The pullback of the Finsler norm on the tangent bundle of $Ham(\mathbf{CP}^{n-1}, \omega)$ is a bi-invariant Finsler norm on $SU(n)$ defined by the operator norm $\|A\| = \sup_{\|x\|=1} \|A(x)\|$ on the Lie algebra of $SU(n)$.

If G is an infinite-dimensional Lie group it is not clear how to describe Δ_{l+1}^G as a geometric object. Nevertheless, as it will be shown below, in the case when $G = \widetilde{Ham}(M, \omega)$ one can use a certain estimate on $\Upsilon_l(\mathcal{C}_1, \dots, \mathcal{C}_l)$ to get a generalization of the ABW inequalities. The same sort of estimate also leads to some interesting applications concerning the geodesics on $Ham(M, \omega)$ with respect to the Hofer metric. The estimate on $\Upsilon_l(\mathcal{C}_1, \dots, \mathcal{C}_l)$ will be given in terms of actions of some periodic orbits of Hamiltonian flows related to $\mathcal{C}_1, \dots, \mathcal{C}_l$.

1.3 The main result

Before stating the result we briefly recall basic definitions.

Hamiltonian functions, vector fields and symplectomorphisms.

Let (M, ω) be a closed symplectic manifold. A function $h : S^1 \times M \rightarrow \mathbf{R}$ (called *Hamiltonian function* or simply *Hamiltonian*) defines a (time-dependent) *Hamiltonian vector field* X_h by the formula

$$dh^t(\cdot) = \omega(X_{h^t}, \cdot), \quad (1)$$

where $h^t = h(t, \cdot)$, and the formula holds pointwise on M for every $t \in S^1$.

A *Hamiltonian symplectomorphism* of (M, ω) is a diffeomorphism of M which can be represented as the time-1 map of the flow of a (time-dependent) Hamiltonian vector field X_h . In such a case we say that the Hamiltonian symplectomorphism is *generated* by the Hamiltonian h .

Normalization condition. All the Hamiltonians $h : S^1 \times M \rightarrow \mathbf{R}$ in the paper will be assumed to be normalized so that $\int_M h^t \omega^n = 0$, for any $t \in S^1$. Such a Hamiltonian will be called a *normalized Hamiltonian*.

The group $Ham(M, \omega)$ and the Hofer metric on it.

Hamiltonian symplectomorphisms of (M, ω) form a group $Ham(M, \omega)$ which can be viewed as an infinite-dimensional Lie group: it can be equipped with the structure of an infinite-dimensional manifold so that the group product and taking the inverse of an element become smooth operations [35]. The Lie algebra \mathcal{H} of $Ham(M, \omega)$ can be identified with the (Poisson) Lie algebra of all functions on M with the zero mean value. The norm $\|h\| = \max_M |h|$ on \mathcal{H} defines a bi-invariant Finsler norm on the tangent bundle of the group $Ham(M, \omega)$. It is a deep result of symplectic topology (see [17], [29], [22]) that the Finsler norm on the tangent bundle of the group leads to a genuine bi-invariant Finsler metric ρ on the group itself. This metric is called *Hofer metric* (the original metric introduced by Hofer was actually defined by the norm $\|h\| = \max_M h - \min_M h$ and is equivalent to the metric we use). The Hofer metric on $Ham(M, \omega)$ lifts to a bi-invariant Finsler pseudo-metric on the universal cover $\widetilde{Ham}(M, \omega)$ of the group $Ham(M, \omega)$. The Finsler pseudo-metric on $\widetilde{Ham}(M, \omega)$ defines the corresponding function Υ_l on l -tuples of conjugacy classes in $\widetilde{Ham}(M, \omega)$. The Hamiltonian flow of X_h over a period of time from 0 to 1, viewed as a path in $Ham(M, \omega)$ starting

at Id , determines an element in $\widetilde{Ham}(M, \omega)$. We will denote the conjugacy class of such an element in $\widetilde{Ham}(M, \omega)$ by \mathcal{C}_h .

Action functional.

Consider the set of pairs (γ, f) , where $\gamma : S^1 \rightarrow M$ is a contractible curve and $f : D^2 \rightarrow M$, is a disk spanning the curve γ , i.e. $f|_{\partial D^2} = \gamma$. Introduce an equivalence relation on the set of such pairs as follows: two pairs (γ, f) and (γ, f') are called *equivalent* if the connected sum $f \# f'$ represents a torsion class in $H_2(M, \mathbf{Z})$. Denote by $\mathcal{P}(h)$ the space of equivalence classes $\hat{\gamma} = [\gamma, f]$ of pairs (γ, f) , where $\gamma : S^1 \rightarrow M$ is a contractible time-1 periodic trajectory of the Hamiltonian flow of h . The theorems proving the Arnold's conjecture [13], [23] imply that any Hamiltonian symplectomorphism of M has a closed contractible periodic orbit and therefore $\mathcal{P}(h)$ is always non-empty.

Define Π as the group of spherical homology classes of M , i.e. as the image of the Hurewicz homomorphism $\pi_2(M) \rightarrow H_2(M, \mathbf{Z})/\text{Tors}$. The group Π acts on $\mathcal{P}(h)$ by the formula:

$$A : [\gamma, f] \mapsto [\gamma, A \# f].$$

For an element $\hat{\gamma} = [\gamma, f] \in \mathcal{P}(h)$ define its *action*:

$$\mathcal{A}_h(\hat{\gamma}) = - \int_D f^* \omega - \int_0^1 h(t, \gamma(t)) dt.$$

Given l Hamiltonians $H = (H_1, \dots, H_l)$ on M we denote by $\mathcal{P}(H)$ the set of equivalence classes $\hat{\gamma} = [\hat{\gamma}_1, \dots, \hat{\gamma}_l]$, where $\hat{\gamma}_i = [\gamma_i, f_i] \in \mathcal{P}(H_i)$ and the equivalence relation is given by

$$[\hat{\gamma}_1, \dots, \hat{\gamma}_l] \sim [A_1 \# \hat{\gamma}_1, \dots, A_l \# \hat{\gamma}_l],$$

whenever $A_i \in \Pi$ and $A_1 + \dots + A_l$ is a torsion class. The group Π acts on $\mathcal{P}(H)$:

$$A : [\hat{\gamma}_1, \dots, \hat{\gamma}_l] \mapsto [A \# \hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_l].$$

Set $\mathcal{A}_H(\hat{\gamma}) = \mathcal{A}_{H_1}(\hat{\gamma}_1) + \dots + \mathcal{A}_{H_l}(\hat{\gamma}_l)$ for $\hat{\gamma} \in \mathcal{P}(H)$.

Moduli spaces $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$.

Let Σ be a complex Riemann surface of genus 0 with l cylindrical ends $\Sigma_i \cong [0, +\infty) \times S^1$, $i = 1, \dots, l$. Consider the trivial fibration $\Sigma \times M \rightarrow \Sigma$

and let pr_M denote the natural projection of $\Sigma \times M$ on M . Let $H = (H_1, \dots, H_l)$ be some (time-dependent) Hamiltonians and let $\hat{\gamma}_i = [\gamma_i, f_i] \in \mathcal{P}(H_i)$, $i = 1, \dots, l$. Set $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_l)$. We will be interested in a certain class $\mathcal{T}(H)$ of almost complex structures on $\Sigma \times M$ depending on H – see Section 4 for the definition. Given an almost complex structure $\tilde{J} \in \mathcal{T}(H)$ we consider the \tilde{J} -holomorphic sections $u : \Sigma \rightarrow \Sigma \times M$ such that the ends of the surface $pr_M \circ u(\Sigma) \subset M$ converge at infinity to the periodic orbits $\gamma_1, \dots, \gamma_l$ and such that the curve $pr_M \circ u(\Sigma) \subset M$ capped off with the discs $f_1(D^2), \dots, f_l(D^2)$ forms a closed surface representing a torsion integral homology class in M . The space of such pseudo-holomorphic sections u will be denoted by $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$.

Given (time-dependent) normalized Hamiltonians $H = (H_1, \dots, H_l)$ on M we will introduce a certain number $size(H)$ (see Definition 3.7.3) and consider a certain family of subsets $\{\mathcal{T}_\tau(H)\} \subset \mathcal{T}(H)$, $0 < \tau < size(H)$ – see Definition 4.0.6. We will say that $\hat{\gamma} \in \mathcal{P}(H)$ is *durable* if there exists a sequence $\{\tau_k\} \nearrow size(H)$ such that all the spaces $\mathcal{M}(\hat{\gamma}, H, \tilde{J}_{\tau_k})$ are non-empty.

Now we state our main result.

Theorem 1.3.1 *Let $H = (H_1, \dots, H_l)$ be some (time-dependent) normalized Hamiltonians on M and let $\mathcal{C}_H = (\mathcal{C}_{H_1}, \dots, \mathcal{C}_{H_l})$ be the corresponding conjugacy classes in $\widetilde{Ham}(M, \omega)$. Then for any durable $\hat{\gamma} \in \mathcal{P}(H)$ one has*

$$\Upsilon_l(\mathcal{C}_H) \geq \mathcal{A}_H(\hat{\gamma}).$$

The inequalities from Theorem 1.3.1 will be called *action inequalities*. Each inequality boils down to the fact that the integral of a certain symplectic form Ω_τ on $\Sigma \times M$ over a \tilde{J} -holomorphic curve is non-negative as soon as \tilde{J} is compatible with the symplectic form Ω_τ (i.e. $\Omega_\tau(\cdot, \tilde{J}\cdot)$ defines a Riemannian metric on $\Sigma \times M$).

The fact that certain moduli spaces $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ are non-empty can be checked in some cases by considering the structure of the “pair-of-pants” product in the Floer cohomology on the level of Floer cochains – the zero-dimensional spaces $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ are instrumental in defining such a product (see Section 6.3).

Corollary 1.3.2 *Suppose that $\hat{\gamma} \in \mathcal{P}(H)$ is durable and $\Upsilon_l(\mathcal{C}_H) = 0$. Then $\mathcal{A}_H(\hat{\gamma}) \leq 0$.*

This is a generalization of the ABW inequalities for the case of the Lie group $G = \widetilde{Ham}(M, \omega)$. However, unlike in [1], [4], we do not know whether the inequalities from Corollary 1.3.2 provide a complete list of inequalities defining Δ_l^G .

1.4 An overview of the applications

In Section 2.4 we will demonstrate how the ABW inequalities for $SU(n)$ can be recovered from action inequalities if one considers the natural group actions of $SU(n)$ on all the complex Grassmannians $Gr(r, n)$, $1 \leq r \leq n-1$, and if the elements of $SU(n)$ are viewed as Hamiltonian symplectomorphisms of the Grassmannians. This approach also indicates what should be the generalization of the ABW inequalities for other Lie groups (see Section 2.4).

We will also apply Theorem 1.3.1 to the study of geodesics in the group $Ham(M, \omega)$, extending previous results of Lalonde and McDuff (see Section 2.3).

1.5 K-area

The key observation which provides the connection between the function Υ_l coming from the Hofer geometry on $\widetilde{Ham}(M, \omega)$ and the pseudo-holomorphic curves is that $\Upsilon_l(\mathcal{C}_1, \dots, \mathcal{C}_l)$ can be interpreted in terms of a certain *K-area*. Roughly speaking, K-area is the inverse of a quantity obtained by fixing a class of connections on some fibration and taking infimum of a C^0 -norm of the curvature tensors of connections in the fixed class (assuming that we have some prefixed metrics on the base and on the fiber used to measure the norm of a curvature tensor). The notion of K-area and its applications to symplectic topology were first introduced by M.Gromov in his seminal paper [16]. Other remarkable applications of K-area to symplectic topology were discovered later by L.Polterovich (see [30], [31], [32]). He studied Hamiltonian fibrations over S^2 and, in particular, found a close connection between the following objects:

- the K-area of a Hamiltonian fibration over S^2 (where the bi-invariant Hofer metric on the group of Hamiltonian symplectomorphisms of the fiber is used for the measurements in the definition of the K-area);
- the Hofer length of the clutching loop for the Hamiltonian fibration (i.e. the loop in the group of Hamiltonian symplectomorphisms of the fiber used

to construct the fibration over S^2 from the trivial fibrations over the two hemispheres).

L.Polterovich also found a way to use K-area as a tool to extract an estimate on the Hofer length of the clutching loop from the fact that a symplectic form on the total space of the fibration integrates non-negatively over a pseudo-holomorphic section of the fibration (see [30]).

In this paper we extend Polterovich's methods and results to fibrations over an oriented surface Σ with boundary. We will be especially interested in the case when the genus of Σ is zero and the fibration is Hamiltonian, i.e. its typical fiber is a closed symplectic manifold (M, ω) and the structural group is $G = \text{Ham}(M, \omega)$. (For another interesting development of the Polterovich's ideas in the similar spirit but in a different setup see [2]).

A G -fibration with a connected fiber over a surface Σ as above is always trivial. Therefore in order to have an interesting quantity one needs to define K-area using the connections whose holonomies along the boundary components of Σ lie in some fixed conjugacy classes $(\mathcal{C}_1, \dots, \mathcal{C}_l)$. Thus K-area becomes a function that associates a number to each tuple $(\mathcal{C}_1, \dots, \mathcal{C}_l)$ of conjugacy classes in G , and as it will be shown in the paper, in the case when the genus of Σ is zero, this function is actually the inverse of Υ_l . (In particular, if G is finite-dimensional then the tuples from Δ_l^G are exactly all possible tuples of conjugacy classes of holonomies of flat connections over $\partial\Sigma$ on a trivial fibration $G \times \Sigma \rightarrow \Sigma$).

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2 Applications of Theorem 1.3.1

2.1 Preliminaries

Before stating the applications of Theorem 1.3.1 we recall the basic facts about quantum and Morse cohomology and define a special class of time-independent Hamiltonians, called *slow Hamiltonians*.

2.1.1 Strongly semi-positive symplectic manifolds

We say that an almost complex structure J on (M, ω) is *compatible* with ω if $\omega(\cdot, J\cdot)$ is a Riemannian metric on M . Almost complex structures compatible with ω form a contractible space.

For the technical reasons coming from the theory of pseudo-holomorphic curves we assume from now on that (M^{2n}, ω) is *strongly semi-positive*, although in view of the recent developments (see [13], [23], [24]) it is likely that this assumption can be removed. Namely, a closed symplectic manifold (M^{2n}, ω) is called *strongly semi-positive*, if for every $A \in \pi_2(M)$ one has

$$2 - n \leq c_1(A) < 0 \implies \omega(A) \leq 0, \quad (2)$$

where c_1 is the first Chern class of the tangent bundle T^*M equipped with an almost complex structure J . In [45] such manifolds were said to satisfy the “Assumption W^+ ”.

Recall that M is called *semi-positive* [28] or *weakly monotone* [18] if in (2) one replaces the inequality $2 - n \leq c_1(A) \leq 0$ by $3 - n \leq c_1(A) \leq 0$ thus weakening the condition. Strong semi-positivity of M means that for $m = 0, 1, 2, 3$ a generic m -parametric family of ω -compatible almost complex structures on (M, ω) does not contain an almost complex structure J for which there exist J -holomorphic spheres with a negative Chern number. In the case of usual (not strong) semi-positivity this holds only for $m = 0, 1, 2$ and since we would like to consider 2-parametric families of almost complex structures and 1-parametric deformations of such families (as in [45]) we ask the manifold to be strongly semi-positive.

The class of strongly semi-positive symplectic manifolds includes closed symplectic surfaces, symplectic tori, complex projective spaces, complex Grassmannians, complex flag manifolds and other interesting objects.

2.1.2 Quantum cohomology

Let (M, ω) , be a closed connected strongly semi-positive symplectic manifold. We will briefly recall the necessary basic facts about quantum cohomology of (M, ω) .

First, we define the coefficient ring for the quantum cohomology. Namely, from the group Π one builds an appropriate Novikov graded ring Λ_ω over \mathbf{Q} : an element of Λ_ω is a formal sum

$$\lambda = \sum_{A \in \Pi} \lambda_A e^{2\pi i A}$$

with rational coefficients $\lambda_A \in \mathbf{Q}$ which satisfies the condition

$$\#\{A \in \Pi \mid \lambda_A \neq 0, \omega(A) \leq c\} < \infty$$

for every $c > 0$. The natural multiplication makes Λ_ω a ring. This ring has a natural grading defined by $\deg(e^{2\pi i A}) = 2c_1(A)$. The subring of elements of degree zero will be denoted by $\Lambda_0 \subset \Lambda_\omega$.

The *quantum cohomology group* of M is defined as a graded tensor product $QH^*(M) = H^*(M) \otimes \Lambda_\omega$, where $H^*(M)$ denotes the quotient $H^*(M, \mathbf{Z})/Tors$.

Given homology classes $A \in \Pi$, and $\alpha_j \in H_{i_j}(M)$, $j = 1, \dots, l$, satisfying the condition

$$i_1 + \dots + i_l = 2n(l-1) - 2c_1(A)$$

one can define the *Gromov-Witten number* $(\alpha_1, \dots, \alpha_l)_A$ (see [54], [37], [38]). The Gromov-Witten numbers define the ring structure on the quantum cohomology group. Namely, let $a_1, \dots, a_l \in H^*(M)$ and let $PD(a_1), \dots, PD(a_{l-1})$ be the corresponding Poincaré-dual homology classes. The multiplication, or *the quantum product* on the quantum cohomology group is defined by the formula:

$$a_1 * \dots * a_{l-1} = \sum_{A \in \Pi} (a_1 * \dots * a_{l-1})_A e^{2\pi i A},$$

where the class $(a_1 * \dots * a_{l-1})_A \in H^{(2n-i_1)+\dots+(2n-i_{l-1})-2c_1(A)}(M)$ has to satisfy the condition

$$\langle (a_1 * \dots * a_{l-1})_A, \alpha_l \rangle = (PD(a_1), \dots, PD(a_{l-1}), \alpha_l)_A$$

for any homology class $\alpha_l \in H_*(M)$. It was proved in [37] that the quantum product is associative. The class $(a_1 * \dots * a_l)_0$ represents the usual cup-product: $(a_1 * \dots * a_l)_0 = a_1 \cup \dots \cup a_l$. The cohomology class $\mathbf{1} \in H^0(M)$ Poincaré-dual to the fundamental class $[M]$ is the unit element in $QH^*(M)$.

2.1.3 Morse homology and cohomology

Here we briefly the definitions of Morse homology and cohomology (see e.g. [41] for details). Fix a Riemannian metric on M^{2n} . Suppose that h is a Morse function on M^{2n} , and moreover, it is a Morse-Smale function with respect to the metric. In such a case one can define the *Morse chain complex* $C_*(h)$ of the function h over the graded coefficient ring Λ_ω as a free graded module over Λ_ω generated by the critical points of h graded by their Morse indices. The differential in such a complex is defined by means of counting downward gradient trajectories of h with respect to the metric that connect the critical points of neighboring indices (see [41] for details). Similarly one can define the dual cochain complex $C^*(h)$ of $C_*(h)$. The homology of the chain complexes $C_*(h)$ and $C^*(h)$ are called, respectively, *the Morse homology and cohomology* of h . The Morse homology (resp. cohomology) of a Morse function is canonically isomorphic to the singular homology (resp. cohomology) of M . The tautological identification $C^k(-h) \cong C_{2n-k}(h)$ leads to the Poincaré isomorphism between the Morse cohomology of $-h$ and the Morse homology of h .

2.1.4 Critical points of functions and multiplicative identities in quantum cohomology

Definition 2.1.1 (cf. [34]) Fix a Riemannian metric on the closed symplectic manifold M^{2n} . Let h be a Morse-Smale function on M with respect to the metric. Let $Crit(h)$ denote the set of all critical points of h , or the set of all generators of the Morse complex $C_*(h)$ over Λ_ω . whose differential is denoted by ∂ . Identify the Morse homology of h with the singular homology of M . We say that a critical point y_i of h *homologically essential* for a homology class $\alpha \in H_*(M, \Lambda_\omega)$ if α , viewed as a class in $H_*(C_*(h))$, does not belong to the image of $i_{\mathcal{K}_*} : H_*(\mathcal{K}_*) \rightarrow H_*(C_*(h))$ for any subcomplex $\mathcal{K}_* \subset \text{Span}(Crit(h) \setminus \{y_i\})$. Equivalently, one can say that y_i is homologically essential for a homology class α if it enters with a non-zero coefficient into any chain in $C_*(h)$ representing α .

Similar questions concerning the necessity of critical points in a chain representing a homology class in the context of generating functions were studied by C.Viterbo in [52].

Example 2.1.2 (cf. [34]) Suppose that y is a unique point of *global* maximum for a function h which is Morse-Smale with respect to some Riemannian metric on M . Let $(C_*(h), \partial)$ be the Morse complex of h . Then we claim that y is homologically essential for the fundamental class $[M] \in H_*(C_*(h))$. Indeed, the subspace $\text{Span}(Crit(h) \setminus \{y\}) \subset C_*(h)$ is ∂ -invariant and the chain complex $(\text{Span}(Crit(h) \setminus \{y\}), \partial)$ is nothing else but the Morse complex for the function h on an *open* manifold $M \setminus y$. But $H_{2n}(M) = \Lambda_\omega$ and $H_{2n}(M \setminus y) = 0$ which proves the claim.

Let H_1, \dots, H_l be Morse Hamiltonians on (M, ω) . Let $A \in \Pi$ and let z^i , $i = 1, \dots, l$, be a critical point of H_i . We say that $\hat{\gamma} = [\hat{\gamma}_1, \dots, \hat{\gamma}_l] \in \mathcal{P}(H)$ is *associated with* z^1, \dots, z^l, A if $\hat{\gamma}_i \in \mathcal{P}(H_i)$, $i = 1, \dots, l-1$, is formed by the pair of constant maps to z^i and $\hat{\gamma}_l \in \mathcal{P}(H_l)$ is formed by a constant map $S^1 \rightarrow z^l$ to z^l and a smooth 2-sphere attached to z^l which realizing the spherical homology class A .

Convention: identification of Morse and singular (co)homology.

From now on, given Morse Hamiltonians H_1, \dots, H_l we will always identify the singular homology and cohomology of M with, respectively, the Morse homology and cohomology of $-H_i$, $i = 1, \dots, l$.

Definition 2.1.3 Suppose that for some cohomology classes $c_1, \dots, c_{l-1} \in H^*(M, \mathbf{Q})$ one has

$$c_1 * \dots * c_{l-1} = \sum_{B \in \Pi} c_B e^{2\pi i B}, \quad (3)$$

where $c_B \in H^*(M, \mathbf{Q})$. Let z^i , be a critical point of H_i , $i = 1, \dots, l$, and let $A \in \Pi$.

We say that $\hat{\gamma}$ associated with z^1, \dots, z^l, A is *involved* in the identity (3) if the following conditions hold:

- The point z^l , viewed as a critical point of $-H_l$, is homologically essential for the rational (singular) homology class Poincaré-dual to c_A , viewed, under our identification convention, as a homology class of the Morse complex $C_*(-H_l)$.
- For each $i = 1, \dots, l-1$ there exists a basis $\alpha_1^i, \dots, \alpha_N^i$, $N = \dim H_*(M, \mathbf{Q})$, of $H_*(M, \mathbf{Q})$ over \mathbf{Q} such that exactly one basic element $\alpha_{j(i)}^i$ satisfies the following two conditions:

(A) z_i , viewed as a critical point of $-H_i$, is essential for $\alpha_{j(i)}^i$, viewed, under our identification convention, as a homology class of the Morse complex $C_*(-H_i)$.

(B) $c_i(\alpha_{j(i)}^i) \neq 0$.

Such a basis $\{\alpha_j^i\}$ of $H_*(M, \mathbf{Q})$ will be called *i-friendly*.

Example 2.1.4 Consider the element $\mathbf{1} \in QH^0(M)$ and the identity

$$\mathbf{1} * \dots * \mathbf{1} = \mathbf{1} \quad (4)$$

with $l - 1$ factors in the left-hand side. Suppose that z_i , $i = 1, \dots, l - 1$, is a unique point of global maximum of H_i and z_l is a unique point of global minimum of H_l . Let $\hat{\gamma}_i$, $i = 1, \dots, l$, be formed by a pair of constant maps into z_i . Then, as one easily sees from Example 2.1.2, $\hat{\gamma}$ is involved in the identity (4). In this particular case any basis in homology is *i-friendly* for any $i = 1, \dots, l - 1$, since it has to include exactly one generator from $H_0(M, \mathbf{Q}) \cong \mathbf{Q}$.

Example 2.1.5 Suppose that H_1, \dots, H_l are perfect Morse functions on M , i.e. the differentials in their Morse chain complexes over the integers, and hence in the Morse complexes $C_*(-H_i)$, $C^*(-H_i)$, $i = 1, \dots, l$, over Λ_ω , are zero. Denote the critical points of $-H_i$, $i = 1, \dots, l$, by z_1^i, \dots, z_N^i , $N = \dim H_*(M, \mathbf{Q})$. Identify $QH_*(M)$, as before with the Morse homology of $C_*(-H_i)$, $i = 1, \dots, l$ with the coefficients in Λ_ω .

For a fixed i the points z_j^i , $j = 1, \dots, N$, viewed as the homology classes in $H_*(C_*(-H_i))$ form a basis of $H_*(C_*(-H_i)) \cong QH_*(M)$ over Λ_ω . Denote the dual basis in the cohomology $H_*(C^*(-H_i)) \cong QH^*(M)$ by Z_j^i , $j = 1, \dots, N$, i.e. Z_j^i takes value 1 on z_j^i and zero on any other critical point of $-H_i$.

The cohomology classes Y_j^i , $j = 1, \dots, N$, Poincaré-dual to z_j^i , form another basis in $H_*(C^*(-H_i)) \cong QH^*(M)$, and the homology classes y_j^i , $j = 1, \dots, N$, Poincaré-dual to Z_j^i , form another basis in the homology $H_*(C_*(-H_i)) \cong QH_*(M)$ dual to the basis Y_j^i , $j = 1, \dots, N$, in the cohomology. In such a case we will say that the *homology class y_j^i is Poincaré-dual to z_j^i* .

Now let $A \in \Pi$ and let $z_{j(i)}^i$, $i = 1, \dots, l$, be a critical point of $-H_i$ of Morse index $m(i)$, so that

$$\sum_{i=1}^l (2n - m(i)) - 2c_1(A) = 2n.$$

Let $\hat{\gamma}_i$, $i = 1, \dots, l-1$, be formed by a pair of constant maps into $z_{j(i)}^i$ and let $\hat{\gamma}_l \in \mathcal{P}(H_l)$ be formed by a constant map $S^1 \rightarrow z_{j(l)}^l$ and a smooth 2-sphere attached to $z_{j(l)}^l$ that realizes the homology class A (here $z_{j(i)}^i$, $i = 1, \dots, l$, is viewed as a critical point of H_i).

Write the quantum product of the cohomology classes $Z_{j(1)}^1, \dots, Z_{j(l-1)}^{l-1}$ as

$$Z_{j(1)}^1 * \dots * Z_{j(l-1)}^{l-1} = \sum_{B \in \Pi} \lambda_B e^{2\pi i B} = \sum_{B \in \Pi, 1 \leq j \leq N} \lambda_{B,j} Y_j^l e^{2\pi i B}, \quad (5)$$

where each $\lambda_B \in H^*(M, \Lambda_\omega)$ is decomposed along the basis $\{Y_j^l\}$, $j = 1, \dots, N$, with the coefficients $\lambda_{B,j} \in \mathbf{Z}$. According to the definition of quantum multiplication, the coefficient $\lambda_{A,j(l)}$ equals to the Gromov-Witten number $(y_{j(1)}^1, \dots, y_{j(l)}^l)_A$.

Then $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_l)$ is involved in the identity (5) if and only if the Gromov-Witten number $(y_{j(1)}^1, \dots, y_{j(l)}^l)_A$, or the coefficient $\lambda_{A,j}$ at the term $Y_{j(l)}^l e^{2\pi i A}$ in (5) is non-zero.

In this case for each $i = 1, \dots, l-1$ the homology classes z_j^i , $j = 1, \dots, N$, form an i -friendly basis of $H_*(M, \mathbf{Q})$.

Observe that this example can be immediately generalized to the situation when not all the differentials in the Morse complexes $C_*(-H_i)$, $i = 1, \dots, l$, are zero but only the ones surrounding the level $m(i)$:

$$\partial_{m(i)+1} : C_{m(i)+1}(-H_i) \rightarrow C_{m(i)}(-H_i)$$

and

$$\partial_{m(i)} : C_{m(i)}(-H_i) \rightarrow C_{m(i)-1}(-H_i).$$

2.1.5 Slow Hamiltonians

Definition 2.1.6 A Hamiltonian h on a closed symplectic manifold (M, ω) is called *slow* if it satisfies the following conditions:

(A) h is time-independent;

(B) the Hamiltonian flow of h has only constant contractible periodic trajectories of period less or equal than 1;

(C) the Hessian of h at any of its critical points does not have an imaginary eigenvalue $i\lambda$ with $\lambda \geq 2\pi$.

A (local) result of Siegel and Moser [49] (also see [25]) shows that a generic Hamiltonian satisfying the conditions (A) and (B) of Definition 2.1.6 also satisfies the condition (C).

As an example of a slow Hamiltonian one can pick any sufficiently C^2 -small function on M (see e.g. [19]).

2.2 Time-independent Hamiltonians: the main result

The following result is essentially based on Theorem 1.3.1 and on the fact that for a slow Morse Hamiltonian the Floer complex can be identified with the Morse complex and the pair-of-pants product on the dual Morse complex descends to the quantum product on the Morse cohomology – see Section 6.3.

Theorem 2.2.1 *Let (M, ω) be strongly semi-positive. Let $H = (H_1, \dots, H_l)$ be normalized slow Morse Hamiltonians. Let z^i , $i = 1, \dots, l$, be some critical points of H_i and let $A \in \Pi$ so that $\hat{\gamma} \in \mathcal{P}(H)$, associated with z^1, \dots, z^l, A , is involved in some identity (3) in the quantum cohomology of (M, ω) .*

Then

$$\Upsilon_l(\mathcal{C}_H) \geq \mathcal{A}_H(\hat{\gamma}) = -H_1(z^1) - \dots - H_l(z^l) - \omega(A).$$

Using Example 2.1.5 one immediately gets from Theorem 2.2.1 the following

Corollary 2.2.2 *Let $H = (H_1, \dots, H_l)$ be normalized slow Hamiltonians which are perfect Morse functions on a strongly semi-positive (M, ω) . Let $A \in \Pi$ and let z^i , $i = 1, \dots, l$, be a critical point of $-H_i$ of Morse index $m(i)$, viewed under the identification $H_*(C_*(-H_i)) \cong QH_*(M)$ as an integral homology class of M of degree $m(i)$, so that*

$$m(1) + \dots + m(l) = 2n(l-1) - 2c_1(A).$$

Let $y^i \in H_{2n-m(i)}(M)$, $i = 1, \dots, l$, be the homology class Poincaré-dual to z^i and suppose that $(y^1, \dots, y^l)_A \neq 0$.

Then

$$\Upsilon_l(\mathcal{C}_H) \geq \mathcal{A}_H(\hat{\gamma}) = -H_1(z^1) - \dots - H_l(z^l) - \omega(A).$$

2.3 Application: geodesics in $Ham(M, \omega)$.

Assume as before that (M, ω) is a closed strongly semi-positive symplectic manifold.

Suppose that $H_1, H_2 : M \rightarrow \mathbf{R}$ are normalized slow Hamiltonians that generate, respectively, Hamiltonian symplectomorphisms φ_{H_1} and φ_{H_2} whose conjugacy classes in $Ham(M, \omega)$ are denoted by $[\varphi_{H_1}]$ and $[\varphi_{H_2}]$. Denote by \mathcal{C}_{H_1} and \mathcal{C}_{H_2} the conjugacy classes in the universal cover of $Ham(M, \omega)$ corresponding, respectively, to the Hamiltonian flows of H_1 and H_2 over the interval of time from 0 to 1. Let $\{a\}$ be a homotopy class of paths in $Ham(M, \omega)$ connecting $[\varphi_{H_1}]$ and $[\varphi_{H_2}]$ whose lifts to the universal cover of $Ham(M, \omega)$ connect \mathcal{C}_{H_1} and \mathcal{C}_{H_2} . Denote by $\rho_{\{a\}}([\varphi_{H_1}], [\varphi_{H_2}])$ the infimum of lengths of paths connecting $[\varphi_{H_1}]$ and $[\varphi_{H_2}]$ from the homotopy class $\{a\}$.

Theorem 2.3.1 *In the notation as above one has*

$$\rho_{\{a\}}([\varphi_{H_1}], [\varphi_{H_2}]) \geq \max(|\max_M H_1 - \max_M H_2|, |\min_M H_1 - \min_M H_2|).$$

Given two Hamiltonian flows $\{f_t\}, \{g_t\}$, generated, respectively, by Hamiltonians F, G , the composition $\{h_t = f_t g_t\}$ is a Hamiltonian flow generated by the Hamiltonian $H(t, x) = F(t, x) + G(t, f_t^{-1}(x))$. Also given a Hamiltonian symplectomorphism φ_H generated by a time-independent Hamiltonian $H : M \rightarrow \mathbf{R}$ one has that $\phi^{-1} \circ \varphi_H \circ \phi$ can be generated by the Hamiltonian function $H \circ \phi$ for any $\phi \in Ham(M, \omega)$. This immediately imposes an estimate from above on $\rho_{\{a\}}([\varphi_{H_1}], [\varphi_{H_2}])$:

$$\rho_{\{a\}}([\varphi_{H_1}], [\varphi_{H_2}]) \leq \inf_{\phi \in Ham(M, \omega)} \|H_1 - H_2 \circ \phi\|,$$

where $\|\cdot\|$ is the norm $\|h\| = \max_M |h|$ on the Lie algebra of $Ham(M, \omega)$ defining the Hofer metric on the group.

Let H be a normalized slow Hamiltonian. Considering $H_1 = \tau_1 H, H_2 = \tau_2 H$ for any $0 < \tau_1 < \tau_2 \leq 1$, and applying Theorem 2.3.1 to the Hamiltonians H_1, H_2 and then to the Hamiltonians $-H_1, -H_2$ generating the symplectomorphisms $\varphi_{-H_1} = \varphi_{H_1}^{-1}, \varphi_{-H_2} = \varphi_{H_2}^{-1}$, one readily obtains the following corollary.

Corollary 2.3.2 *For any $0 \leq \tau_1, \tau_2 \leq 1$ the path $\{\varphi_H^\tau\}, \tau \in [\tau_1, \tau_2]$, is globally length-minimizing in its homotopy class in $Ham(M, \omega)$.*

This result has been previously proved by F. Lalonde and D. McDuff for symplectic manifolds which are either 2-dimensional or weakly exact (see [21], Theorem 5.4). An extension of this result to the case of a general closed symplectic manifold has also been the subject of a recent independent work by D. McDuff and J. Slimowitz [25] (also see [50]).

Let us now state a result similar to Theorem 2.3.1 for the case when $l > 2$.

Theorem 2.3.3 *Suppose that $H = (H_1, \dots, H_l)$ are normalized slow Hamiltonians and let $\mathcal{C}_H = (\mathcal{C}_{H_1}, \dots, \mathcal{C}_{H_l})$ be the corresponding conjugacy classes in $\widetilde{Ham}(M, \omega)$. Then*

$$\Upsilon_l(\mathcal{C}_H) \geq -\max_M H_1 - \dots - \max_M H_{l-1} - \min_M H_l. \quad (6)$$

As an easy application of Theorem 2.3.1 consider the case when $M = S^2$ with the standard symplectic form ω normalized so that $\int_{S^2} \omega = 1$. Let φ_1 and φ_2 be the linear rotations of S^2 around the same axis by angles $0 \leq \xi_1 \leq \xi_2 \leq \pi$ respectively. Clearly, $\varphi_1, \varphi_2 \in Ham(S^2, \omega)$. In this case we can actually compute $\rho_{\{a\}}([\varphi_1], [\varphi_2])$ for all possible homotopy classes $\{a\}$: there are four such homotopy classes. Indeed, according to a result of Gromov [15] the subgroup $SO(3) \subset Ham(S^2, \omega)$ is a deformational retract of $Ham(S^2, \omega)$. Hence $\pi_1(Ham(S^2, \omega)) = \pi_1(SO(3)) = \mathbf{Z}_2$ and the path in $SO(3)$ defining the full 2π twist of S^2 around some axis represents the generator of \mathbf{Z}_2 . This shows that there only two homotopy classes of paths connecting each of the elements φ_1 and φ_2 with the identity in $Ham(S^2, \omega)$ (they are represented by “clockwise” and “counterclockwise” rotations around the oriented axis). Therefore we have four possible homotopy classes $\{a\}$. All the Hamiltonians can be easily computed: φ_i is generated by an autonomous Hamiltonian $H_i : S^2 \rightarrow \mathbf{R}$ which has only two critical points with critical values $\max H_i = \zeta_i/2$ and $\min H_i = -\zeta_i/2$.

Applying Theorem 2.3.1 we recover the result of F. Lalonde and D. McDuff (see [21], Corollary 1.10) concerning geodesics in $Ham(S^2, \omega)$ (also see [30] for further developments).

Corollary 2.3.4 ([21]) *The distance $\rho(\varphi_1, \varphi_2)$ between φ_1 and φ_2 in the group $Ham(S^2)$ equals $\frac{\zeta_2 - \zeta_1}{2}$.*

As another application of Theorem 1.3.1 we give an estimate from below on Υ_3 in the case of the symplectic two-dimensional torus $M = \mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. This estimate is different from the one in Theorem 2.3.3. Observe that in the case of $M = \mathbf{T}^2$ there is no need to pass to the universal cover of $Ham(\mathbf{T}^2, \omega)$ since this group is already simply connected (see e.g. [34]).

Let $H_1, H_2, H_3 : \mathbf{T}^2 \rightarrow \mathbf{R}$ be normalized slow Hamiltonians which are perfect Morse functions. Denote by $[\varphi_{H_1}], [\varphi_{H_2}], [\varphi_{H_3}]$ the conjugacy classes of the Hamiltonian symplectomorphisms generated, respectively, by H_1, H_2, H_3 .

Denote by z_3 the point of global maximum of H_3 and denote by z_1, z_2 the critical points of index 1 of, respectively, H_1 and H_2 so that z_1 and z_2 correspond to the homology classes α_1, α_2 that generate $H_1(\mathbf{T}^2)$.

Theorem 2.3.5 *In the notation as above*

$$\Upsilon_3([\varphi_{H_1}], [\varphi_{H_2}], [\varphi_{H_3}]) \geq -H_1(z_1) - H_2(z_2) - H_3(z_3).$$

2.4 Application: Grassmannians, ABW inequalities

In this section we discuss the relation between the ABW and the action inequalities.

Consider the Grassmannian $Gr(r, n)$ of complex r -planes in \mathbf{C}^n . The Grassmannian can be viewed as the result of symplectic reduction for the Hamiltonian action of $U(r)$ (by multiplication from the right) on the space of complex $r \times n$ matrices. The symplectic structure ω on $Gr(r, n)$ that we choose is constructed as the one induced on the symplectic reduction by the standard symplectic structure on the space \mathbf{C}^{nr} of complex $r \times n$ matrices. The natural complex structure on $Gr(r, n)$ is compatible with ω and we assume that ω is normalized so that the cohomology class $[\omega]$ takes value 1 on the generator of $H_2(Gr(r, n), \mathbf{Z}) = \Pi \cong \mathbf{Z}$ realized by a complex sphere. One quickly checks that $(Gr(r, n), \omega)$ is a strongly semi-positive symplectic manifold.

If one fixes a complete flag $\{0\} = F_0 \subset \dots \subset F_n = \mathbf{C}^n$ then to each subset $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$ corresponds a Schubert variety

$$\{W_I \in Gr(r, n) \mid \dim(W_I \cap F_{i_j}^\perp) \geq j, j = 1, \dots, r\}$$

representing an integer homology class σ_I which does not actually depend on the choice of the flag. Given homology classes $\sigma_{I_1}, \dots, \sigma_{I_l} \in H_*(Gr(r, n))$

and a class $d \in \mathbf{Z} \cong H_2(Gr(r, n), \mathbf{Z})$ one can define the corresponding Gromov-Witten invariant $(\sigma_{I_1}, \dots, \sigma_{I_l})_d$. The Gromov-Witten invariants mentioned above determine the multiplicative structure of the quantum cohomology ring of $(Gr(r, n), \omega)$ (see [5], [6], [47], [55]).

The group $SU(n)$ naturally acts on $Gr(r, n)$. This action is Hamiltonian and it produces the homomorphism $SU(n) \rightarrow Ham(Gr(r, n), \omega)$ with the *finite* kernel \mathbf{Z}_n . Since the kernel is finite the homomorphism is a covering map from G to its image which is an embedded submanifold of $Ham(M, \omega)$. This allows to pull back the Finsler norm on $T_*Ham(Gr(r, n), \omega)$ (defining the Hofer metric on the group $Ham(Gr(r, n), \omega)$) to an induced Finsler norm on $T_*SU(n)$. This Finsler norm leads to a Finsler metric on $SU(n)$ and by means of this metric one defines the function Υ_l on l -tuples of conjugacy classes in $SU(n)$. Also consider the pullback of the same Finsler norm from $T_*Ham(Gr(r, n), \omega)$ to $T_*\widetilde{Ham}(Gr(r, n), \omega)$ and consider the function $\Upsilon_l^{(r)}$ on l -tuples of conjugacy classes in $\widetilde{Ham}(Gr(r, n), \omega)$ defined by means of the corresponding Finsler pseudo-metric on the group.

Choose the fundamental domain \mathfrak{U} in the Cartan subalgebra of $su(n)$ to be defined by the equations:

$$\alpha_1 + \dots + \alpha_n = 0, \alpha_1 \geq \dots \geq \alpha_n \geq \alpha_1 - 1,$$

so that \mathcal{C}_α , $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathfrak{U}$, is the conjugacy class of an element of $SU(n)$ with the eigenvalues $e^{2\pi i \alpha_1}, \dots, e^{2\pi i \alpha_n}$.

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathfrak{U}$ consider the 1-parametric subgroup $diag(e^{2\pi i \alpha_1 t}, \dots, e^{2\pi i \alpha_n t}) \subset SU(n)$ of diagonal matrices. The action of this 1-parametric subgroup on $Gr(r, n)$ produces a Hamiltonian flow $\{\varphi_{H_\alpha}^t\}$ for some time-independent Hamiltonian $H_\alpha : Gr(r, n) \rightarrow \mathbf{R}$ with mean value zero. Thus H_α is normalized. Denote by $\mathcal{C}_\alpha^{(r)} := \mathcal{C}_{H_\alpha}$ the conjugacy class in $\widetilde{Ham}(Gr(r, n), \omega)$ containing the element defined by the flow.

One can check that for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathfrak{U}$ such that $\alpha_i \neq \alpha_j$, $1 \leq i, j \leq n$, the Hamiltonian H_α is a Morse function. Such a Hamiltonian H_α has two crucial properties (see Section 8):

- 1) H_α is a slow Hamiltonian. This is basically due to the fact that \mathfrak{U} lies in the domain of injectivity of the exponential map.
- 2) H_α is a perfect Morse function. Indeed, the critical points of H_α are in one-to-one correspondence with the invariant complex r -dimensional subspaces of the matrix $diag(e^{2\pi i \alpha_1 t}, \dots, e^{2\pi i \alpha_n t})$ or, equivalently, with the subsets $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$, or, equivalently, with the free generators σ_I of the homology group of $Gr(r, n)$.

These properties of H_α will allow us to identify the Floer chain complex corresponding to H_α and the Morse chain complex corresponding to $-H_\alpha$ so that the quantum product of cohomology classes of $(Gr(r, n), \omega)$ gets identified with the Floer product of the same classes realized by the Morse cohomology classes of $-H_\alpha$ (see Section 6.3).

Now let $\zeta^i = (\zeta_1^i, \dots, \zeta_n^i) \in \mathfrak{U}$, $i = 1, \dots, l$, $\zeta = (\zeta^1, \dots, \zeta^l) \in \mathfrak{U}^l$ and let $\mathcal{C}_\zeta = (\mathcal{C}_{\zeta^1}, \dots, \mathcal{C}_{\zeta^l})$ be the corresponding conjugacy classes in $SU(n)$. For the conjugacy classes $\mathcal{C}_\zeta = (\mathcal{C}_{\zeta^1}, \dots, \mathcal{C}_{\zeta^l})$ in $SU(n)$ denote by $\mathcal{C}_\zeta^{(r)} = (\mathcal{C}_{\zeta^1}^{(r)}, \dots, \mathcal{C}_{\zeta^l}^{(r)})$ the corresponding conjugacy classes in $\widetilde{Ham}(Gr(r, n), \omega)$.

Theorem 2.4.1 *Suppose $(\sigma_{I_1}, \dots, \sigma_{I_l})_d \neq 0$ in $Gr(r, n)$. Then*

$$\Upsilon_l(\mathcal{C}_\zeta) \geq \Upsilon_l^{(r)}(\mathcal{C}_\zeta^{(r)}) \geq \sum_{j=1}^l \sum_{i \in I_j} \zeta_i^j - d.$$

As a corollary we obtain the original ABW inequalities for $SU(n)$.

Corollary 2.4.2 (cf. [1],[4]) *Suppose $\Upsilon_l(\mathcal{C}_\zeta) = 0$. Then for any r , $1 \leq r \leq n-1$, and any I_j , $|I_j| = r$, $j = 1, \dots, l$, such that $(\sigma_{I_1}, \dots, \sigma_{I_r})_d \neq 0$ in $Gr(r, n)$ one has $\sum_{j=1}^l \sum_{i \in I_j} \zeta_i^j \leq d$.*

In fact, according to [1],[4] the converse is also true: if for any r , $1 \leq r \leq n-1$, and any I_j , $|I_j| = r$, $j = 1, \dots, l$, such that $(\sigma_{I_1}, \dots, \sigma_{I_r})_d \neq 0$ in $Gr(r, n)$ one has $\sum_{j=1}^l \sum_{i \in I_j} \zeta_i^j \leq d$ then $\Upsilon_l(\mathcal{C}_\zeta) = 0$. (To see it observe that all the ABW inequalities for all l can be deduced from the inequalities for $l = 2$, in which case the statement can be checked easily).

Example 2.4.3 Consider the case $G = SU(2)$, $l = 3$. The conjugacy class of a matrix from $SU(2)$ with eigenvalues $e^{\pm 2\pi i \zeta}$, $0 \leq \zeta \leq 1/2$, is completely determined by the real number ζ . Then Δ_3^G is polytope of maximal dimension which lies inside the cube $[0, 1/2] \times [0, 1/2] \times [0, 1/2]$ in \mathbf{R}^3 . The inequalities defining Δ_3^G and Υ_3 can be computed in this case directly by elementary methods (cf. [20]) and the result, of course, matches Theorem 2.4.1 and Corollary 2.4.2. The polytope Δ_3^G is a tetrahedron defined by the equations

$$\zeta^1 + \zeta^2 + \zeta^3 \leq 1,$$

$$\begin{aligned}\zeta^1 &\leq \zeta^2 + \zeta^3, \\ \zeta^2 &\leq \zeta^1 + \zeta^3, \\ \zeta^3 &\leq \zeta^1 + \zeta^2,\end{aligned}$$

corresponding to its four faces. The function Υ_3 on a triple of conjugacy classes (viewed each as a point in $[0, 1/2]$) is given by the formula:

$$\Upsilon_3(\zeta^1, \zeta^2, \zeta^3) = \max\{0, \zeta^3 - \min(\zeta^1 + \zeta^2, 1 - \zeta^1 - \zeta^2)\}.$$

Considering this formula for $(\zeta^1, \zeta^2, \zeta^3)$ close to zero one gets that the inequalities in Theorem 2.4.1 may turn into equalities.

The proofs of the results above indicate how one can possibly describe the convex polytope Δ_l^G for a compact semi-simple connected and simply-connected Lie group G other than $SU(n)$. Let us briefly sketch how this can be done. Given such a Lie group G with the Lie algebra \mathfrak{g} one should consider all its compact Kähler homogeneous spaces (these are in one-to-one correspondence with subsets of the set of simple roots of G – see [46], [53]). Since G is semi-simple, one has $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $H^2(\mathfrak{g}) = 0$. Therefore the natural action of G on any such homogeneous space (M, ω) is Hamiltonian. (As F.Lalonde has pointed out to me, this is true even if G is not semi-simple: using the flux homomorphism one can easily deduce it from the fact that G is just compact and simply-connected). The kernel of this action is the center Z of G which is finite because G is semi-simple. Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ any element of G acts on (M, ω) as a Hamiltonian symplectomorphism generated by a Hamiltonian (see Definition 2.1.6) that can be represented as the Poisson bracket of some other Hamiltonians and therefore is normalized. Thus the Hamiltonian action of G on (M, ω) induces an inclusion of \mathfrak{g} into the Lie algebra of functions on M with mean value zero. Pick a fundamental domain $\mathfrak{U} \subset \mathfrak{t} \subset \mathfrak{g}$ containing zero and lying in the domain of injectivity of the exponential map. Now given an element $v \in \mathfrak{U}$ consider the Hamiltonian flow on (M, ω) induced by the action of the 1-parametric subgroup $\{e^{\tau v}\}_{0 \leq \tau \leq 1}$. For a generic v this Hamiltonian flow is generated by a normalized Morse Hamiltonian function H_v . As in the case of $SU(n)$ the function H_v satisfies two crucial properties:

- 1) H_v is a slow Hamiltonian. As in the case of $SU(n)$ this is based on the fact that \mathfrak{U} lies in the domain of injectivity of the exponential map.
- 2) H_v is a perfect Morse function. This can be checked using the cell decomposition for M which is the analogue of the Schubert cell decomposition for complex Grassmannians (see [7], [8]).

The actions of the constant periodic trajectories of the Hamiltonian flow generated by H_v depend only on the conjugacy class of $e^{\tau v}$ in G and completely determine that class. Then to any non-zero Gromov-Witten number $(\alpha_1, \dots, \alpha_l)_A$ for generators $\alpha_1, \dots, \alpha_l$ from the basis of $H_*(M)$ given by the cell decomposition one associates the corresponding action inequality for tuples of conjugacy classes in the universal cover of $Ham(M, \omega)$. This would in turn provide a generalized ABW inequality that would have to be satisfied by any tuple belonging to Δ_l^G . Considering all the non-zero Gromov-Witten numbers for all compact Kähler homogeneous spaces (M, ω) of G one should get a complete set of inequalities defining the convex polytope Δ_l^G .

Recently, in [51], C.Teleman and C.Woodward, using the method which is completely different from the one presented above but rather extends the algebraic geometry approach from [1], have found a nice set of inequalities defining the polytope Δ_l^G (where, as before, G is a compact complex semi-simple connected and simply-connected Lie group). Their inequalities are written in intrinsic terms of the structure of the Lie algebra of G .

Remark 2.4.4 Observe that if G is a finite-dimensional connected Lie group then one has an analogue of Theorem 2.4.1 for any homomorphism $G \rightarrow Ham(M, \omega)$ with a finite kernel. As before this circumstance allows us to pull back the Finsler norm on the tangent bundle of $Ham(M, \omega)$ to the tangent bundle of G , where it determines a genuine Finsler metric on the group G itself.

3 Fibrations over a surface with boundary, K-area and weak coupling

3.1 Our favorite surface Σ

Let Σ be a compact connected oriented Riemann surface of genus 0 with $l \geq 1$ boundary components: $\partial\Sigma = T_1 \sqcup \dots \sqcup T_l$. Fix a volume form Ω on Σ so that $\int_{\Sigma} \Omega = 1$. According to the Moser's theorem [27], any two such volume forms coinciding near $\partial\Sigma$ can be mapped into each other by a diffeomorphism of Σ .

The orientation of Σ determines an orientation on each T_i (rotate an outward normal vector to Σ by 90 degrees counterclockwise to get the positive direction on $\partial\Sigma$).

3.2 Fibrations, connections, curvatures and holonomies

Let G be a connected Lie group whose tangent bundle is equipped with a bi-invariant Finsler norm that defines a pseudo-metric ρ on G . Identify the Lie algebra \mathfrak{g} of G with the space of right-invariant vector fields on G . Below we will not use any results from the Lie theory and thus all our considerations will hold even if G is an infinite-dimensional Lie group. Suppose that G acts effectively on a connected manifold F (i.e. one has a monomorphism $G \rightarrow \text{Diff}(F)$). The case which is most important for us is when $F = (M, \omega)$ is a symplectic manifold and $G = \text{Ham}(M, \omega)$.

Consider the trivial G -bundle $\pi : P \rightarrow \Sigma$, with the fiber F . Let us consider G -connections on the bundle $P \rightarrow \Sigma$, i.e. the connections whose parallel transports belong to the structural group G . Let L^{∇} denote the curvature of a connection ∇ on the bundle $\pi : P \rightarrow \Sigma$. If the fiber $\pi^{-1}(x)$ is identified with F then to a pair of vectors $v, w \in T_x \Sigma$ the curvature tensor associates an element $L^{\nabla}(v, w) \in \mathfrak{g}$. Here we use the fact that G acts effectively on F . If no identification of $\pi^{-1}(x)$ with F is fixed then $L^{\nabla}(v, w) \in \mathfrak{g}$ is defined up to the adjoint action of G on \mathfrak{g} . Thus if $\|\cdot\|$ is the bi-invariant Finsler norm on \mathfrak{g} defining our Finsler norm on T_*G then $\|L^{\nabla}(v, w)\|$ does not depend on the identification of $\pi^{-1}(x)$ with F .

Definition 3.2.1 We define $\|L^{\nabla}\|$ as

$$\|L^{\nabla}\| = \max_{v, w} \frac{\|L^{\nabla}(v, w)\|}{|\Omega(v, w)|},$$

where the maximum is taken over all pairs $(v, w) \in T_*\Sigma \times T_*\Sigma$ such that $\Omega(v, w) \neq 0$.

3.3 The definition of K-area

Given a G -connection ∇ on $\pi : P \rightarrow \Sigma$ the holonomy of ∇ along a loop based at $x \in \Sigma$ can be viewed as an element of G acting on F provided that the bundle is trivialized over x . If the trivialization of the bundle is allowed to vary then the holonomy is defined up to conjugation in the group G . Observe that the action of the gauge group does not change $\|L^\nabla\|$.

Now let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_l)$ be some conjugacy classes in G .

Definition 3.3.1 Let $\mathcal{L}(\mathcal{C})$ denote the set of connections ∇ on $P \rightarrow \Sigma$ which are flat over a neighborhood of $\partial\Sigma$ and such that for any $i = 1, \dots, l$ and the holonomy of ∇ along the oriented boundary component $T_i \subset \partial\Sigma$ lies in \mathcal{C}_i .

Definition 3.3.2 The number $0 < K\text{-area}(\mathcal{C}) \leq +\infty$ is defined as

$$K\text{-area}(\mathcal{C}) = \sup_{\nabla \in \mathcal{L}(\mathcal{C})} \|L^\nabla\|^{-1}. \quad (7)$$

Obviously, since none of the boundary components of Σ is preferred over the others, the quantity $K\text{-area}(\mathcal{C}_1, \dots, \mathcal{C}_l)$ does not depend on the order of the conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_l$.

3.4 The relation between K-area and Υ_l

The principal relation between K-area and Υ_l is expressed in the following theorem.

Theorem 3.4.1

$$\Upsilon_l(\mathcal{C}) = \frac{1}{K\text{-area}(\mathcal{C})}.$$

(If $K\text{-area}$ is infinite its inverse is assumed to be zero).

We will prove Theorem 3.4.1 in a stronger form (see Theorem 3.6.1) but in order to formulate it we need to introduce more definitions.

3.5 Systems of paths and their homotopy classes

The quantity Υ_l can be defined in a different way as follows.

Definition 3.5.1 A *system of paths* $a = (a_1, \dots, a_l)$ is a tuple of some smooth paths $a_1, \dots, a_l : [0, 1] \rightarrow G$ such that

$$a_1(0) \cdot \dots \cdot a_l(0) = Id.$$

The *length*(a) of a system of paths a is defined as the sum of lengths of the paths that form the system, where the length of a path is measured with respect to the fixed Finsler pseudo-metric on G .

Definition 3.5.2 Let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_l)$ be some conjugacy classes in G . We define $\mathcal{G}(\mathcal{C})$ as the set of all systems of paths (a_1, \dots, a_l) such that $a_i(1) \in \mathcal{C}_i$, $i = 1, \dots, l$.

Proposition 3.5.3

$$\Upsilon_l(\mathcal{C}) = \inf_{a \in \mathcal{G}(\mathcal{C})} \text{length}(a). \quad (8)$$

We will prove Proposition 3.5.3 in Section 10.

In the case when the group G is not simply connected the space $\mathcal{G}(\mathcal{C})$ might have more than one connected component: there might be systems of paths in $\mathcal{G}(\mathcal{C})$ that are not homotopic to each other. Denote such a homotopy class of a system of paths a by $[a]$. and denote the corresponding connected component of $\mathcal{G}(\mathcal{C})$ by $\mathcal{G}_{[a]}(\mathcal{C})$.

Definition 3.5.4 For a homotopy class $[a]$ of systems of paths from $\mathcal{G}(\mathcal{C})$ define $\Upsilon_{l,[a]}(\mathcal{C})$ by taking in (8) the infimum only over the systems of paths from $\mathcal{G}_{[a]}(\mathcal{C})$.

Thus $\Upsilon_l(\mathcal{C}) = \inf_{[a]} \Upsilon_{l,[a]}(\mathcal{C})$.

Just as the space $\mathcal{G}(\mathcal{C})$ of systems of paths may not be connected, the space of connections $\mathcal{L}(\mathcal{C})$ on the trivial principal G -bundle $P \rightarrow \Sigma$ also might have many connected components, i.e. different connections from $\mathcal{L}(\mathcal{C})$ might not be homotopic to each other in $\mathcal{L}(\mathcal{C})$. Such a homotopy class of a connection ∇ will be denoted by $[\nabla]$ and the corresponding connected component of $\mathcal{L}(\mathcal{C})$ will be denoted as $\mathcal{L}_{[\nabla]}(\mathcal{C})$. If one fixes a homotopy class $[\nabla]$ one can define $K\text{-area}_{[\nabla]}(\mathcal{C})$ by taking the supremum in (7) over $\mathcal{L}_{[\nabla]}(\mathcal{C})$.

3.6 A stronger version of Theorem 3.4.1

Now we are able to state a stronger version of Theorem 3.4.1.

Theorem 3.6.1 *To a homotopy class $[\nabla]$ of connections from $\mathcal{L}(\mathcal{C})$ one can naturally associate in a surjective way a homotopy class $[[\nabla]] = [a]([\nabla])$ of systems of paths from $\mathcal{G}(\mathcal{C})$ so that*

$$\Upsilon_{l, [[\nabla]]}(\mathcal{C}) = \frac{1}{K\text{-area}_{[\nabla]}(\mathcal{C})}.$$

Theorem 3.6.1 will be proven in Section 11. As it will easily follow from the proof the correspondence $[\nabla] \rightarrow [[\nabla]]$ between homotopy classes of connections and homotopy classes of systems of paths satisfies the relation below.

For each of the conjugacy classes \mathcal{C}_i , $i = 1, \dots, l$, consider all paths in our group G connecting \mathcal{C}_i with the identity. Pick a homotopy class $\{c_i\}$ of such paths. The homotopy class $\{c_i\}$ determines a certain conjugacy class $\tilde{\mathcal{C}}_i$ in the universal cover \tilde{G} of G .

Given the homotopy classes $\{c\} = (\{c_1\}, \dots, \{c_l\})$ as above and a system of paths $a = (a_1, \dots, a_l)$ from $\mathcal{G}(\mathcal{C})$ complete each path $a_i : [0, 1] \rightarrow G$ with $a_i(1) \in \mathcal{C}_i$ by a curve connecting $a_i(1)$ with the identity and representing a path from $\{c_i\}$. In this way one gets a curve $\tilde{a}_i : [0, 1] \rightarrow G$ such that $\tilde{a}_i(1) = Id$, $\tilde{a}_i(0) = a_i(0)$. Since, according to the definition of systems of paths, $a_1(0) \dots a_l(0) = Id$, the pointwise group product $t \rightarrow \tilde{a}_1(t) \dots \tilde{a}_l(t)$, $0 \leq t \leq 1$, of paths $\tilde{a}_1, \dots, \tilde{a}_l$ represents a loop in G based at the identity. The homotopy class of this loop depends only on $\{c\}$ and on the homotopy class $[a]$ of the system of paths. If the loop is contractible we say that $[a]$ fits with $\{c\}$.

Denote by $\mathcal{L}_{\{c\}}^{fit}(\mathcal{C})$ the set of all the connections ∇ in $\mathcal{L}(\mathcal{C})$ such that $[[\nabla]]$ fits with $\{c\}$. The property of lying in $\mathcal{L}_{\{c\}}^{fit}(\mathcal{C})$ depends only on the homotopy class $[\nabla]$ of a connection ∇ .

Proposition 3.6.2 *Let $\{c\}$, \mathcal{C} and $\tilde{\mathcal{C}}$ be as above. Consider the pseudo-metric on \tilde{G} defined by the pullback of the Finsler norm from T_*G and let $\tilde{\Upsilon}_l(\tilde{\mathcal{C}})$ be defined by means of this pseudo-metric. Then*

$$\tilde{\Upsilon}_l(\tilde{\mathcal{C}}) = \inf_{[[\nabla]]} \Upsilon_{l, [[\nabla]]}(\mathcal{C}) = \inf_{[\nabla]} \frac{1}{K\text{-area}_{[\nabla]}(\mathcal{C})},$$

where the infimums are taken over all homotopy classes $[\nabla]$ of connections lying in $\mathcal{L}_{\{c\}}^{fit}(\mathcal{C})$.

In particular, this means that in order to study $\tilde{\Upsilon}_l(\tilde{\mathcal{C}})$ by means of K-area we only need to consider connections from $\mathcal{L}_{\{c\}}^{fit}$.

3.7 Connections on Hamiltonian fibrations and weak coupling

Consider now the specific case of the trivial G -bundle $P \rightarrow \Sigma$ with the fiber $F = (M, \omega)$, where $G = Ham(M, \omega)$ and Σ is the compact surface of genus 0 with l boundary components as above.

Let us briefly recall the following basic definitions (see [14] for details).

Definition 3.7.1 A closed 2-form $\tilde{\omega}$ on the total space P of the bundle $P \rightarrow \Sigma$ is called *fiber compatible* if its restriction on each fiber of $P \rightarrow \Sigma$ is ω .

Let us trivialize the bundle $P = \Sigma \times M \rightarrow M$ and let $pr_M : P \rightarrow M$ be the natural projection. The *weak coupling construction* [14] prescribes that for any fiber compatible form $\tilde{\omega}$ that coincides with $pr_M^* \omega$ near ∂P and for sufficiently small $\varepsilon > 0$ there exists a smooth family of closed 2-forms $\{\Omega_\tau\}$, $\tau \in [0, \varepsilon]$, on P with the following properties:

- (i) $\Omega_0 = \pi^* \Omega$, where π is the projection $\pi : P \rightarrow \Sigma$ and Ω is the fixed symplectic form on the surface Σ ;
- (ii) $[\Omega_\tau] = \tau[\tilde{\omega}] + [\pi^* \Omega]$, where the cohomology classes are taken in $H^2(P, \partial P)$;
- (iii) the restriction of Ω_τ on each fiber of π is a multiple of the symplectic form on that fiber;
- (iv) Ω_τ is symplectic for $\tau \in (0, \varepsilon]$.

Definition 3.7.2 We define *size*($\tilde{\omega}$) as the supremum of all ε that admit a family $\{\Omega_\tau\}$, $\tau \in [0, \varepsilon]$, satisfying the properties (i)-(iv) listed above.

Any fiber compatible form $\tilde{\omega}$ defines a connection ∇ on $\pi : P \rightarrow \Sigma$ and, conversely, any Hamiltonian connection on $P \rightarrow \Sigma$ can be defined by a unique fiber compatible 2-form $\tilde{\omega}_\nabla$ such that the 2-form on Σ obtained from $\tilde{\omega}_\nabla^{n+1}$ by fiber integration is 0. (see e.g. [14]).

Recall that the Lie algebra of $Ham(M, \omega)$, which is the algebra of all (globally) Hamiltonian vector fields on (M, ω) , is identified with the space of functions on (M, ω) with the zero mean value (see Section 1.3). Therefore the curvature of ∇ can be viewed as a 2-form associating to each pair $v, w \in T_x \Sigma$ of tangent vectors on the base a normalized Hamiltonian function $H_{v,w}$ on the fiber $\pi^{-1}(x)$. The form $\tilde{\omega}_\nabla$ restricted on the horizontal lifts of vectors $v, w \in T_x \Sigma$ at a point $y \in \pi^{-1}(x)$ coincides with $H_{v,w}(y)$ (see e.g. [14]).

Let $H = (H_1, \dots, H_l)$ be some (time-dependent) normalized Hamiltonians on M . Let $[\varphi_H] = ([\varphi_{H_1}], \dots, [\varphi_{H_l}])$ be the conjugacy classes in $Ham(M, \omega)$ containing the Hamiltonian symplectomorphisms $\varphi_{H_1}, \dots, \varphi_{H_l}$ generated by H_1, \dots, H_l . Also let $\mathcal{C}_H = (\mathcal{C}_{H_1}, \dots, \mathcal{C}_{H_l})$ be the conjugacy classes in $\widehat{Ham}(M, \omega)$ corresponding to $H = (H_1, \dots, H_l)$.

Define $\mathcal{F}([\varphi_H])$ as the set of all the forms $\tilde{\omega}_\nabla$, $\nabla \in \mathcal{L}([\varphi_H])$. Given a homotopy class $[\nabla]$ of connections from $\nabla \in \mathcal{L}([\varphi_H])$ define $\mathcal{F}_{[\nabla]}([\varphi_H])$ as the set of all the forms $\tilde{\omega}_\nabla$, $\nabla \in \mathcal{L}_{[\nabla]}([\varphi_H])$. The Hamiltonian flow generated by H_i , $1 \leq i \leq l$, represents a homotopy class $\{c_i\}$ of paths connecting Id with $[\varphi_{H_i}]$. Let $\{c\} = (\{c_1\}, \dots, \{c_l\})$. In Section 3.6 we defined the set $\mathcal{L}_{\{c\}}^{fit}([\varphi_H])$. Define $\mathcal{F}_H([\varphi_H])$ as the set of all the forms $\tilde{\omega}_\nabla$, $\nabla \in \mathcal{L}_{\{c\}}^{fit}([\varphi_H])$.

Definition 3.7.3 Define the numbers

$$0 < size([\varphi_H]), size_{[\nabla]}([\varphi_H]), size(H) \leq +\infty$$

as

$$\begin{aligned} size([\varphi_H]) &= \sup_{\tilde{\omega} \in \mathcal{F}([\varphi_H])} size(\tilde{\omega}), \\ size_{[\nabla]}([\varphi_H]) &= \sup_{\tilde{\omega} \in \mathcal{F}_{[\nabla]}([\varphi_H])} size(\tilde{\omega}), \\ size(H) &= \sup_{\tilde{\omega} \in \mathcal{F}_H([\varphi_H])} size(\tilde{\omega}). \end{aligned}$$

The following theorem can be proved by exactly the same arguments as the similar theorems in [30], [32].

Theorem 3.7.4 *Let K -area be measured, as before, with respect to the norm $\|h\| = \max_M |h|$ on the Lie algebra of $\text{Ham}(M, \omega)$ (both for the group $\text{Ham}(M, \omega)$ and its universal cover).*

Then

$$K\text{-area}_{[\nabla]}([\varphi_H]) \leq \text{size}_{[\nabla]}([\varphi_H]).$$

for any $[\nabla]$ and, in general,

$$K\text{-area}([\varphi_H]) \leq \text{size}([\varphi_H]).$$

Also

$$K\text{-area}(\mathcal{C}_H) \leq \text{size}(H).$$

4 The classes $\mathcal{T}_\tau(H)$, $\mathcal{T}(H)$, $\mathcal{T}^0(H)$ of almost complex structures and moduli spaces $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$

In this section we will fix a trivialization $P = \Sigma \times M$ of trivial bundle $P \rightarrow \Sigma$ that we considered before. We rescale our surface Σ to present it as a non-compact surface of area 1 with l cylindrical ends. We fix an identification $\Phi_i : [0, +\infty) \times S^1 \rightarrow \Sigma_i$, $1 \leq i \leq l$, of each end $\Sigma_i \subset \Sigma$ with the standard cylinder $[0, +\infty) \times S^1$. Without loss of generality we may assume that the identifications are chosen in such a way that near infinity the conformal structure on the ends gets identified with the standard conformal structure on the cylinder $[0, +\infty) \times S^1$.

If J is an almost complex structure on M compatible with ω and \tilde{J} is an almost complex structure \tilde{J} on $\Sigma \times M$ we say that \tilde{J} is *J-fibered* if the following conditions are fulfilled:

- \tilde{J} preserves the tangent spaces to the fibers of $\pi : \Sigma \times M \rightarrow \Sigma$;
- the restriction of \tilde{J} on any fiber is an almost complex structure compatible with the symplectic form ω on the fiber;
- the restriction of \tilde{J} on any fiber $\pi^{-1}(x)$ for x outside of some compact subset of Σ is J .

Now let $H = (H_1, \dots, H_l)$ be some (time-dependent) Hamiltonians on M . Let us choose some $\hat{\gamma} \in \mathcal{P}(H)$. Let us also pick once and for all a cut-off function $\beta : \mathbf{R} \rightarrow [0, 1]$ such that $\beta(s)$ vanishes for $s \leq \epsilon$ and $\beta(s) = 1$ for $s \geq 1 - \epsilon$ for some small $\epsilon > 0$.

Any section $u : \Sigma \rightarrow P$ by means of the trivialization $P = \Sigma \times M$ induces some maps

$$u_i = u \circ \Phi_i : [0, +\infty) \times S^1 \rightarrow M.$$

Suppose that J is an ω -compatible almost complex structure on M . For each $i = 1, \dots, l$ consider the non-homogeneous Cauchy-Riemann equation

$$\partial_s u_i + J(u_i) \partial_t u_i - \beta(s) \nabla_u H_i(t, u_i) = 0, \quad (9)$$

where gradient is taken with respect to the Riemannian metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ on M . According to [15], 1.4.C, the solutions of such an equation correspond exactly to the pseudo-holomorphic sections of $\pi^{-1}(\Sigma_i) \rightarrow \Sigma_i$ with respect to some unique J -fibered almost complex structure on $\pi^{-1}(\Sigma_i)$ in the following way. Fix an almost complex structure j on Σ which is compatible with the symplectic form Ω and which extends the standard complex structures on the cylinders $\Sigma_i = \Phi_i([0, +\infty) \times S^1)$, $i = 1, \dots, l$.

Definition 4.0.5 Let \tilde{J} be an almost complex structure on P and let $H = (H_1, \dots, H_l)$ be Hamiltonians as above. We shall say that \tilde{J} is *H-compatible* if there exists an almost complex structure $J = J(\tilde{J})$ on M compatible with ω such that the following conditions hold.

- \tilde{J} is J -fibered.
- $\pi \circ \tilde{J} = j \circ \pi$, where $\pi : \Sigma \times M \rightarrow \Sigma$ is the projection.
- For each $i = 1, \dots, l$ there exist some number K_i such that over $\tilde{\Sigma}_i = \Phi_i([K_i, +\infty) \times S^1) \subset \Sigma_i = \Phi_i([0, +\infty) \times S^1)$ the \tilde{J} -holomorphic sections of $\pi^{-1}(\tilde{\Sigma}_i) \rightarrow \tilde{\Sigma}_i$, viewed as maps $u_i : [K_i, +\infty) \times S^1 \rightarrow M$, are exactly the solutions of the non-homogeneous Cauchy-Riemann equation (9) for J .

Let us denote by $\mathcal{T}(H)$ the space of all H -compatible almost complex structures on $\Sigma \times M$. We also define a subset $\mathcal{T}^0(H) \subset \mathcal{T}(H)$ that includes only those $\tilde{J} \in \mathcal{T}(H)$ which for some J are split as $\tilde{J} = j \times J$ over a compact part of Σ that contains $\Sigma \setminus \bigcup_{i=1}^l \Sigma_i$, where $J = J(\tilde{J})$. In such a case we will denote $\tilde{J} = \tilde{J}_{J,H}$.

Let $[\varphi_H] = ([\varphi_{H_1}], \dots, [\varphi_{H_l}])$ be the conjugacy classes in $Ham(M, \omega)$ as before.

Now we define the classes $\mathcal{T}_\tau(H)$ of almost complex structures that are used in the definition of a durable element in $\mathcal{P}(H)$ (see Section 1.3).

Definition 4.0.6 Consider all the families $\{\Omega_{\tilde{\omega}_\nabla, \tau}\}$, that arise from the weak coupling construction associated with $\tilde{\omega}_\nabla$, $\nabla \in \mathcal{L}_{\{c\}}^{fit}([\varphi_H])$ (see Section 3.7). Given a number $\tau_0 \in (0, size(H))$ consider the set \mathcal{Q}_{τ_0} of all the forms $\Omega_{\tilde{\omega}_\nabla, \tau_0}$ from the families $\{\Omega_{\tilde{\omega}_\nabla, \tau}\}$ as above (i.e. we consider only those families which are defined for the value τ_0 of the parameter τ and pick the form $\Omega_{\tilde{\omega}_\nabla, \tau_0}$ from each such family). Denote by $\mathcal{T}_{\tau_0}(H)$ the set of all the almost complex structures in $\mathcal{T}(H)$ which are compatible with some symplectic form \mathcal{Q}_{τ_0} .

Below we also give a precise definition of the moduli spaces $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ involved in the definition of a durable element in $\mathcal{P}(H)$ (see Section 1.3).

Let $\hat{\gamma} = [\hat{\gamma}_1, \dots, \hat{\gamma}_l]$, where $\hat{\gamma}_i = [\gamma_i, f_i] \in \mathcal{P}(H_i)$, $i = 1, \dots, l$. Let $\tilde{J} \in \mathcal{T}(H)$.

Definition 4.0.7 (cf. [28]) Denote by $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ the space of all smooth \tilde{J} -holomorphic sections $u : \Sigma \rightarrow P$ which satisfy the following conditions.

(i) The maps $u_i = u \circ \Phi_i : [0, +\infty) \times S^1 \rightarrow M$ constructed above satisfy

$$\gamma_i(t) = \lim_{s \rightarrow +\infty} u_i(s, t).$$

(ii) The closed surface obtained by capping off $u(\Sigma) \subset M$ with the discs f_1, \dots, f_l (taken with the opposite orientations) represents a torsion homology class in $H_2(M, \mathbf{Z})$.

Looking at the condition (ii) one sees that the space $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ depends only on the equivalence class $\hat{\gamma} = [\hat{\gamma}_1, \dots, \hat{\gamma}_l]$ of the tuple $(\hat{\gamma}_1, \dots, \hat{\gamma}_l)$ – see the definition of the equivalence relation in Section 1.3.

5 Proof of Theorem 1.3.1

Let Υ_l^{Ham} be defined on $Ham(M, \omega)$ using the Hofer metric. Let Σ be the surface with l cylindrical ends, as before. Let $\{c\} = (\{c_1\}, \dots, \{c_l\})$ be as in Section 4.

From Proposition 3.6.2 one gets that

$$\Upsilon_l(\mathcal{C}_H) = \inf_{\nabla} \Upsilon_{l, [[\nabla]]}^{Ham}([\varphi_H]) \quad (10)$$

where the infimum is taken over all the connections $\nabla \in \mathcal{L}_{\{c\}}^{fit}([\varphi_H])$.

It follows from the hypothesis of the theorem that for any τ_0 , $0 < \tau_0 < size(H)$, there exist a connection $\nabla \in \mathcal{L}_{\{c\}}^{fit}([\varphi_H])$, the corresponding 2-form $\tilde{\omega}_{\nabla}$ such that $\tau_0 < size(\tilde{\omega}_{\nabla})$, and a weak coupling deformation $\{\Omega_{\tilde{\omega}_{\nabla}, \tau}\}$, $0 < \tau < size(\tilde{\omega}_{\nabla})$, so that the space $\mathcal{M}(\hat{\gamma}, H, \tilde{J}_{\tau_0})$ is non-empty for some $\tilde{J}_{\tau_0} \in \mathcal{T}_{\tau_0}(H)$ compatible with the symplectic form $\Omega_{\tilde{\omega}_{\nabla}, \tau_0}$.

Pick a map $u \in \mathcal{M}(\hat{\gamma}, H, \tilde{J}_{\tau_0})$. Since \tilde{J}_{τ_0} is compatible with the symplectic form $\Omega_{\tilde{\omega}_{\nabla}, \tau_0}$, one has

$$0 \leq \int_{u(\Sigma)} \Omega_{\tilde{\omega}_{\nabla}, \tau_0} = \int_{u(\Sigma)} \tau_0 \tilde{\omega} + \int_{u(\Sigma)} \pi^* \Omega, \quad (11)$$

where $u(\Sigma)$ is viewed as a surface in P and $\pi : P \rightarrow \Sigma$ is the projection.

Lemma 5.0.8

$$\int_{u(\Sigma)} \tilde{\omega}_{\nabla} = -\mathcal{A}_H(\hat{\gamma}). \quad (12)$$

Postponing the proof of the lemma we first finish the proof of the theorem. Indeed, since the total Ω -area of Σ is 1 one can rewrite (11), in the case when $\mathcal{A}_H(\hat{\gamma}) = -\int_{u(\Sigma)} \tilde{\omega}_{\nabla}$ is positive (otherwise the theorem is trivial) as

$$\tau_0 \geq \frac{1}{\mathcal{A}_H(\hat{\gamma})}$$

Since $\nabla \in \mathcal{L}_{\{c\}}^{fit}([\varphi_H])$ was chosen arbitrarily, τ_0 can be chosen arbitrarily close to $size(H)$ one gets that

$$size(H) \geq \frac{1}{\mathcal{A}_H(\hat{\gamma})}.$$

Using this inequality together with (10), Theorem 3.6.1 and Theorem 3.7.4 one readily obtains the needed result. ■

The proof of Lemma 5.0.8.

This is a purely topological fact – we have already used all the complex properties of u that we needed. Therefore we can rescale Σ back and make it a compact surface with boundary. Then one can extend Φ_i , $i = 1, \dots, l$, to a map $\Phi_i : [0, +\infty] \times S^1 \rightarrow \Sigma$ so that $\Phi_i(+\infty \times t)$, $0 \leq t \leq 1$, parameterizes the boundary component T_i of Σ . The map u restricted on the boundary component T_i of Σ produces the curve $\gamma_i \in M$, $i = 1, \dots, l$. We cap off the boundaries of Σ with some discs D_1, \dots, D_l and get a closed surface $\hat{\Sigma}$.

Given a connection $\nabla \in \mathcal{L}_{\{c\}}^{fit}([\varphi_H])$ on the bundle over the compact surface Σ one can assume without loss of generality that the trivialization of $P \rightarrow \Sigma$ is already adjusted in such a way that for any τ the holonomy of ∇ (taken with respect to the trivialization of $P \rightarrow \Sigma$) over the path $\{t \rightarrow \Phi_i(+\infty \times t)\}_{0 \leq t \leq \tau}$ is the flow $\varphi_{H_i}^\tau$, $i = 1, \dots, l$, of H_i for the time τ .

Then it is not hard to prove the following technical sublemma (see e.g. [44], Section 4.1).

Sublemma 5.0.9 *Over a disc D_i , $i = 1, \dots, l$, one can construct a trivialized bundle $E_i = D_i \times M \rightarrow D_i$ together with its section $U_i : D_i \rightarrow E_i$ and a connection ∇_i with the 2-form $\tilde{\omega}_{\nabla_i}$ on E_i with the following properties.*

- (i) *The trivialized bundle $E_i \rightarrow D_i$ agrees along $\partial D_i = \Delta_i$ with the trivialized bundle $P \rightarrow \Sigma$.*
- (ii) *If by means of the trivialization the section $U_i : D_i \rightarrow E_i$ is viewed as map $U_i : D_i \rightarrow M$ then $U_i(D_i) = f_i$, $U_i(\partial D_i) = \gamma_i$.*
- (iii) *The connection ∇_i $\tilde{\omega}_i$ on E_i smoothly extends the connection ∇ on $P \rightarrow \Sigma$.*
- (iv) $\int_{D_i} U_i^* \tilde{\omega}_{\nabla_i} = \mathcal{A}_{H_i}([\gamma_i, f_i])$.

Note that the definition of action functional in [44] differs from the one we use here by the sign at the term containing the integral over a disc and the Hamiltonians used in [44] actually correspond not to our Hamiltonians $H = (H_1, \dots, H_l)$ but to $\bar{H} = (\bar{H}_1, \dots, \bar{H}_l)$, due to a different sign convention given by (1) so that at the end we get the plus sign in the right-hand side of (iv) above.

The lemma allows us the bundles $E_i \rightarrow D_i$, the maps U_i , and the forms $\tilde{\omega}_{\nabla_i}$, $i = 1, \dots, l$, to $P \rightarrow \Sigma$, u and $\tilde{\omega}_{\nabla}$ respectively. Namely, consider again Σ as a compact surface with boundary and consider a closed surface $\hat{\Sigma}$ obtained by capping off each boundary component of Σ with a disc. We construct a trivialized bundle $\hat{P} = \hat{\Sigma} \times M \rightarrow \hat{\Sigma}$ to which we extend the

section u (viewed now as a map $u : \hat{\Sigma} \rightarrow \hat{P}$) and the 2-form $\tilde{\omega}_{\nabla}$ (viewed now as a form on \hat{P}). Recalling the condition (ii) of Definition 4.0.7, we get

$$0 = \int_{u(\hat{\Sigma})} \tilde{\omega}_{\nabla}$$

and hence, according to Lemma 5.0.9,

$$\int_{u(\Sigma)} \tilde{\omega}_{\nabla} = - \sum_{i=1}^l \int_{U_i(D_i)} \tilde{\omega}_{\nabla} = -\mathcal{A}_H(\hat{\gamma}).$$

This gives us the equality (12) and finishes the proof of the lemma. ■

6 The proof of Theorem 2.2.1

Suppose, as in the hypothesis of Theorem 2.2.1, that for some singular cohomology classes $c_1, \dots, c_{l-1} \in H^*(M, \mathbf{Q})$ one has

$$c_1 * \dots * c_{l-1} = \sum_{B \in \Pi} c_B e^{2\pi i B}, \quad (13)$$

where $c_B \in H^*(M, \mathbf{Q})$.

Theorem 2.2.1 follows immediately from Theorem 1.3.1 and the following proposition proved below.

Proposition 6.0.10 *In the notation as in Definition 2.1.3 suppose that $\hat{\gamma} \in \mathcal{P}(H)$ associated with z^1, \dots, z^l, A is involved in an identity (13). Then $\hat{\gamma} \in \mathcal{P}(H)$ is durable.*

To check that $\hat{\gamma} \in \mathcal{P}(H)$ is durable one needs to study of the multiplicative structure of the *Floer cohomology* of (M, ω) defined below.

6.1 Conley-Zehnder index of a periodic trajectory of a Hamiltonian flow

Let h be a Hamiltonian on M . For an element $\hat{\gamma} = [\gamma, f] \in \mathcal{P}(h)$, one can define its *Conley-Zehnder index* $\mu(\hat{\gamma}) \in \mathbf{Z}$ (see [9]). The Conley-Zehnder index satisfies the property

$$\mu(A \sharp \hat{\gamma}) = \mu(\hat{\gamma}) - 2c_1(A).$$

If h is a slow Morse Hamiltonian and $\hat{\gamma} = [\gamma, f]$ is formed by constant maps into a critical point y of H , then $\mu(\hat{\gamma})$ is equal to the Morse coindex of y (i.e. $2n$ minus the Morse index).

For $H = (H_1, \dots, H_l)$ on M and an element $\hat{\gamma} = [\hat{\gamma}_1, \dots, \hat{\gamma}_l] \in \mathcal{P}(H)$ we define

$$\mu(\hat{\gamma}) = \sum_{i=1}^l \mu(\hat{\gamma}_i).$$

6.2 Floer cohomology

Now we recall some basic facts about Floer cohomology.

Assume that (M, ω) , is a closed connected strongly semi-positive symplectic manifold. The Floer theory shows that for a Hamiltonian $h : S^1 \times M \rightarrow \mathbf{R}$ one can define a kind of “Morse homology” for the action functional \mathcal{A}_h over the ring Λ_ω . The construction goes along the following lines (for the details of the theory in the semi-positive case see [18]). Given a generic (a so called *regular*) pair (h, J) , where h is a (time-dependent) Hamiltonian and J is an almost complex structure compatible with ω , one defines a chain complex $CF_*(h, J)$ of Λ_ω -modules. Here $CF_*(h, J)$ is a free graded module over the graded ring Λ_ω generated by the critical points of \mathcal{A}_h which are time-1 contractible periodic orbits of the Hamiltonian flow of h . The grading of such an orbit is given by the Conley-Zehnder index. The differential ∂ in the complex is defined similarly to the finite-dimensional Morse homology by means of counting solutions of an appropriate Cauchy-Riemann equation that represent “the gradient trajectories” connecting critical points of \mathcal{A}_h of neighboring indices (see e.g. [18] for details). One can show that $\partial^2 = 0$ (see [18]). The homology group $HF_*(h, J)$ of the chain complex $CF_*(h, J)$ is called *the Floer homology group* (it is actually a module over Λ_ω). Given two different (regular) pairs (h_α, J_α) and (h_β, J_β) there exists a natural isomorphism $I_*^{\beta\alpha} : HF_*(h_\alpha, J_\alpha) \rightarrow HF_*(h_\beta, J_\beta)$.

Assume now that h is a slow Morse Hamiltonian. Let J be an almost complex structure on M compatible with ω so that h is a Morse-Smale function with respect to the Riemannian metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. The most crucial property of slow Morse Hamiltonians is that for any such J the pair (h, J) is regular and the Floer chain complex $CF_*(h, J)$ can be identified with the Morse chain complex $C_*(-h, g)$ (see [40]). The seemingly strange combination of signs is due to our choice of the signs in the definition of action functional \mathcal{A}_h so that the downward gradient flow of \mathcal{A}_h corresponds to the upward gradient flow of h .

The appropriate map $I_*^{\beta\alpha}$ provides a natural grading-preserving isomorphism between $HF_*(h, J)$ and the homology of the chain complex $C_*(-h, g)$ which leads to a natural isomorphism

$$\mathcal{SF}_h : H_*(M) \otimes \Lambda_\omega \rightarrow HF_*(h, J) \quad (14)$$

such that for any two regular pairs (h_α, J_α) and (h_β, J_β) , with h_α, h_β being slow Morse Hamiltonians, one has $I_*^{\beta\alpha} \circ \mathcal{SF}_{h_\alpha} = \mathcal{SF}_{h_\beta}$.

In a similar fashion, given a Hamiltonian $h : S^1 \times M \rightarrow \mathbf{R}$, one can define the Floer cochain complex $CF^*(h, J) = \text{Hom}(CF_*(h, J), \Lambda_0)$ whose

cohomology $HF^*(h, J)$ is called *the Floer cohomology group*. There exists a natural Poincaré isomorphism $HF^k(h, J) \cong HF_{2n-k}(\bar{h}, J)$, where $\bar{h}(t, x) = -h(-t, x)$. If h is a slow Morse Hamiltonian one can identify the Morse and the Floer cohomology in the same fashion as homology.

6.3 The pair-of-pants product in Floer cohomology

We are going to state the basic facts concerning the pair-of-pants product in Floer cohomology in our specific situation.

Let $H = (H_1, \dots, H_l)$ be our slow Morse Hamiltonians. Let J be an almost complex structure on (M, ω) such that H_1, \dots, H_l are Morse-Smale functions with respect to the metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. Given $0 < \tau < \text{size}(H)$ denote by $\mathcal{T}_J(H)$ the set of all $\tilde{J} \in \mathcal{T}(H)$ such that $J = J(\tilde{J})$. Set $\mathcal{T}_{\tau, J}(H) = \mathcal{T}_{\tau}(H) \cap \mathcal{T}_J(H)$.

The following statement is similar to the one from [28] and [42] although our setup is slightly different from the one used there: here we use almost complex structure from $\mathcal{T}(H)$ while in [28], [42] a smaller class $\mathcal{T}^0(H) \subset \mathcal{T}(H)$ is used.

Proposition 6.3.1 *Assume that (M, ω) is strongly semi-positive. Then for a generic $\tilde{J}_{\tau} \in \mathcal{T}_{\tau, J}(H)$, and any $\hat{\gamma} \in \mathcal{P}(H)$ with $\mu(\hat{\gamma}) = 2n$ the space $\mathcal{M}(\hat{\gamma}, H, \tilde{J}_{\tau})$ is either empty or an oriented compact zero-dimensional manifold.*

Given such a generic \tilde{J}_{τ} we will say that the pair (H, \tilde{J}_{τ}) is *regular*.

Given a regular pair (H, \tilde{J}_{τ}) , $\tilde{J}_{\tau} \in \mathcal{T}_{\tau, J}(H)$, and an element $\hat{\gamma} \in \mathcal{P}(H)$ such that $\mu(\hat{\gamma}) = 2n$ count the curves from the compact zero-dimensional moduli space $\mathcal{M}(\hat{\gamma}, H, \tilde{J}_{\tau})$ with their signs. The resulting Gromov-Witten number will be denoted by $n(\hat{\gamma}, H, \tilde{J}_{\tau})$. Form the sum

$$\theta_{\Sigma, H, \tilde{J}_{\tau}} = \sum_{\hat{\gamma}} n(\hat{\gamma}, H, \tilde{J}_{\tau}) \hat{\gamma}, \quad (15)$$

where the sum is taken over all $\hat{\gamma} \in \mathcal{P}(H)$ such that $\mu(\hat{\gamma}) = 2n$.

The sum in (15) represents an integral chain in the chain complex

$$CF_*(H_1, J) \otimes \dots \otimes CF_*(H_l, J). \quad (16)$$

Roughly speaking, this chain complex is an integral Morse homology complex for the action functional \mathcal{A}_H , $H = (H_1, \dots, H_l)$, and its homology is

equal to $HF_*(H_1, J) \otimes \dots \otimes HF_*(H_l, J)$, where \otimes stands for the graded tensor product over the Novikov ring Λ_0 – see [28] for details.

The following theorem is a slight generalization of the main result from [28]: we use a bigger class of admissible almost complex structures (as in Proposition 6.3.1).

Theorem 6.3.2 *Assume that (M, ω) is strongly semi-positive. For a regular pair (H, \tilde{J}_τ) , $0 < \tau < \text{size}(H)$, $\tilde{J}_\tau \in \mathcal{T}_{\tau, J}(H)$, the chain $\theta_{\Sigma, H, \tilde{J}_\tau}$ defines a cycle in the chain complex (16). The corresponding homology class*

$$\Theta_{H, J} \in HF_*(H_1, J) \otimes \dots \otimes HF_*(H_l, J)$$

is of degree $2n$ and does not depend on $\tilde{J}_\tau \in \mathcal{T}_J(H)$.

By means of the Poincaré duality the homology class $\Theta_{H, J}$ defines the pair-of-pants product

$$\varrho : HF^{i_1}(H_1, J) \otimes \dots \otimes HF^{i_{l-1}}(H_{l-1}, J) \rightarrow HF^{i_1 + \dots + i_{l-1}}(\bar{H}_l, J)$$

on the Floer cohomology which is related to the quantum product in the following way. Given a Hamiltonian function $h : S^1 \times M \rightarrow \mathbf{R}$ there exists a natural isomorphism $\mathcal{QF}_h : QH^(M) \rightarrow HF^*(h, J)$ which intertwines the quantum product on the quantum cohomology with the pair-of-pants product ϱ on the Floer cohomology:*

$$\mathcal{QF}_{\bar{H}_l}(b_1 * \dots * b_{l-1}) = \varrho(\mathcal{QF}_{H_1}(b_1) \otimes \dots \otimes \mathcal{QF}_{H_{l-1}}(b_{l-1})).$$

The following statement, I believe, has been known to the experts but I was unable to find it in a published form.

Proposition 6.3.3 *If h is a slow Morse Hamiltonian then $\mathcal{QF}_h = \mathcal{SF}_h$ where $\mathcal{SF}_h : H^*(M) \otimes \Lambda_\omega \rightarrow HF^*(h, J)$ is the natural isomorphism in cohomology induced by the isomorphism from (14).*

6.4 Proof of Proposition 6.0.10

Let the setup be as in Section 6.3. Using the fact that H_1, \dots, H_l are Morse-Smale functions with respect to the metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ we identify the Floer chain complex $CF_*(H_i, J)$ and the Morse chain complex $C_*(\bar{H}_i)$, $i = 1, \dots, l$. We want to study the properties of the pair-of-pants product on the level of cochains in the (dual) Morse complexes for the slow Morse Hamiltonians.

Pick an arbitrary $0 < \tau < \text{size}(H)$ and a regular pair (H, \tilde{J}) , $\tilde{J}_\tau \in \mathcal{T}_{\tau, J}(H)$. In view of the identification of Floer and Morse complexes and Theorem 6.3.2 the chain

$$\theta_{\Sigma, H, \tilde{J}_\tau} = \sum_{\mu(\hat{\gamma})=2n} n(\hat{\gamma}, H, \tilde{J}_\tau) \hat{\gamma}, \quad (17)$$

can be considered as representing a homology class $\Theta_{H, J}$ in

$$H_*(C_*(\bar{H}_1)) \otimes \dots \otimes H_*(C_*(\bar{H}_l)).$$

(Recall that all the tensor products are taken over $\Lambda_0 \subset \Lambda_\omega$). Theorem 6.3.2 and Proposition 6.3.3 imply that $\Theta_{H, J}$ defines the *quantum* multiplication on $H^*(M, \Lambda_\omega) \cong H_*(C^*(\bar{H}_i))$, $i = 1, \dots, l$.

According to the hypothesis of Proposition 6.0.10, $\hat{\gamma} \in \mathcal{P}(H)$ associated with z^1, \dots, z^l, A is involved in the identity (3) in the quantum cohomology. For each $i = 1, \dots, l-1$ denote by $\{\alpha_j^i\}$, $1 \leq j \leq \dim H_*(M, \mathbf{Q})$, the corresponding i -friendly basis of $H_*(M, \mathbf{Q})$. Denote by z_j^l , $j \in \mathcal{I}$, all the critical points of H_l , where \mathcal{I} is some finite set of indices. Recall that the Morse complexes are taken with coefficients in the ring Λ_ω which contains \mathbf{Q} and therefore has *no torsion*. Thus one can represent the homology class $\Theta_{H, \tilde{J}}$ by a chain as follows:

$$\Theta_{H, J} = \sum \alpha_{j_1}^1 \otimes \dots \otimes \alpha_{j_{l-1}}^{l-1} \otimes [z_{j_l}^l, f_{j_l}^l]. \quad (18)$$

for some $f_{j_l}^l$ such that $[z_{j_l}^l, f_{j_l}^l] \in \mathcal{P}(H_l)$. Here each rational homology class $\alpha_{j_i}^i$, $i = 1, \dots, l-1$, is a generator from the i -friendly basis $\{\alpha_j^i\}$ and is viewed as a homology class of the Morse chain complex of \bar{H}_i over \mathbf{Q} . To get the expression (18) we have incorporated all the coefficients from Λ_ω in $\Theta_{H, J}$ into the last factor of the tensor product and we have expanded a product of a coefficient from Λ_ω with an element of the Λ_ω -module $C_*(\bar{H}_l)$ as a sum of elements from $\mathcal{P}(H_l)$. (The critical points of $\bar{H}_l = -H_l$ generating $C_*(\bar{H}_l)$ are viewed here as critical points of H_l).

Observe that such a representation (18) of $\Theta_{H,J}$ is not unique since different Λ_ω -linear combinations of the critical points z_j^l may represent the same homology class in $H_*(M, \Lambda_\omega)$.

Now we claim that for any τ as above the number $n(\hat{\gamma}, H, \tilde{J}_\tau)$ has to be non-zero and therefore $\hat{\gamma}$ is durable.

Indeed, since the critical point z^l of \bar{H}_l is homologically essential for the rational singular homology class Poincaré-dual to c_A one gets that the decomposition (18) must include a term

$$\alpha_{j_1}^1 \otimes \dots \otimes \alpha_{j_{l-1}}^{l-1} \otimes \hat{\gamma}_l,$$

such that $c_i(\alpha_{j_i}^i) \neq 0$ for any $i = 1, \dots, l-1$. Moreover, it follows from the definition of an i -friendly basis $\{\alpha_j^i\}$ (see Definition 2.1.3) that such a term is unique. Comparing (17) and (18) one sees that the existence and the uniqueness of such a term in (18) imply that $n(\hat{\gamma}, H, \tilde{J}_\tau) \neq 0$. The claim is proven.

This finishes the proof of Proposition 6.0.10 and Theorem 2.2.1. ■

7 Proofs of Theorems 2.3.1, 2.3.3, 2.3.5

In Theorems 2.3.1 and 2.3.3, we can, without loss of generality, consider only the special case when all the slow Hamiltonians are Morse functions that have unique points of global maximum and minimum. Indeed, this condition can be always achieved by a sufficiently C^∞ -small perturbation of the Hamiltonians as functions on M . Thus, since a sufficiently C^∞ -small perturbation of a slow Hamiltonian is again slow, the general case follows from the special case by continuity.

Now Theorem 2.3.3 follows from Theorem 2.2.1 and Example 2.1.4.

Theorem 2.3.1 follows from Theorem 2.3.3 applied in the case $l = 2$ to the pairs $H_1, -H_2$ and $-H_1, H_2$.

To prove Theorem 2.3.5 check, as in Example 2.1.5, that since the Hamiltonians are perfect Morse functions $\hat{\gamma} \in \mathcal{P}(H)$ associated with z_1, z_2, z_3 , $A = 0$ is involved in the identity

$$c_1 \cup c_2 = c_3,$$

where c_1, c_2 form the basis in $H^1(\mathbf{T}^2)$ dual to α_1, α_2 (i.e. $c_i(\alpha_j) = \delta_{ij}$), and c_3 is the generator of $H^2(\mathbf{T}^2)$ which evaluates as 1 on the fundamental class. Then Theorem 2.3.5 follows Theorem 2.2.1. ■

8 Proof of Theorem 2.4.1

Recall that the function Υ_l is on conjugacy classes in $SU(n)$ was defined with respect to the Finsler metric defined by the Finsler norm induced from $T_*Ham(Gr(r, n), \omega)$. Then the first inequality follows by functoriality directly from the definitions.

Now we will prove the second inequality. Let us start with some technical observations. As it was already mentioned, the Grassmannian $Gr(r, n)$ can be viewed as the result of symplectic reduction for the Hamiltonian action of $U(r)$ (by multiplication from the right) on the space \mathbf{C}^{rn} of complex $r \times n$ matrices. Namely the value of the moment map $F : \mathbf{C}^{rn} \rightarrow \mathbf{R}$ of the action on an $r \times n$ matrix A can be written as

$$F(A) = B^*B/2i$$

and the Grassmannian $Gr(r, n)$ can be identified with the quotient $S/U(r)$, where $S = F^{-1}(Id/2i) \subset \mathbf{C}^{rn}$.

Given an $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathfrak{U}$ consider the Hamiltonian action $\{g'_{e^{2\pi it}}\}$, $e^{2\pi it} \in S^1$, of S^1 on \mathbf{C}^n , where an element $e^{2\pi it} \in S^1$ acts on a vector $w = (w_1, \dots, w_n) \in \mathbf{C}^n$ by the formula

$$e^{2\pi it} : (w_1, \dots, w_n) \rightarrow (e^{2\pi it\alpha_1}w_1, \dots, e^{2\pi it\alpha_n}w_n).$$

The action $\{g'_{e^{2\pi it}}\}$ can be viewed as the Hamiltonian flow generated by the autonomous Hamiltonian $H'_\alpha : \mathbf{C}^n \rightarrow \mathbf{R}$ defined by the formula

$$H'_\alpha(w) = -\alpha_1|w_1|^2 - \dots - \alpha_n|w_n|^2.$$

The action $\{g'_{e^{2\pi it}}\}$ induces a Hamiltonian action $\{g_{e^{2\pi it}}\}$, $e^{2\pi it} \in S^1$, of S^1 on $Gr(r, n)$ which can be viewed as the Hamiltonian flow generated by some *normalized* Hamiltonian $H_\alpha : Gr(r, n) \rightarrow \mathbf{R}$.

To study the properties of H_α consider first the Hamiltonian action $\{g''_{e^{2\pi it}}\}$, $e^{2\pi it} \in S^1$, of S^1 on $\mathbf{C}^{rn} = (\mathbf{C}^n)^r$ obtained as the direct product of the actions $\{g'_{e^{2\pi it}}\}$ on the factors \mathbf{C}^n of $(\mathbf{C}^n)^r$. The action $\{g''_{e^{2\pi it}}\}$, can be viewed as the Hamiltonian flow generated by the Hamiltonian $H''_\alpha : (\mathbf{C}^n)^r \rightarrow \mathbf{R}$ which is the direct sum of the Hamiltonians H'_α on the r factors of $(\mathbf{C}^n)^r$.

The Hamiltonian H''_α is constant along the fibers of the action of $U(r)$ on S and thus descends to a function on $Gr(r, n)$. Since $\alpha \in \mathfrak{U}$, one has that $\alpha_1 + \dots + \alpha_n = 0$ and from here one easily deduces that the integral of H''_α over S , and hence the integral of H_α over $Gr(r, n)$ are zero. Thus

the Hamiltonian H''_α restricted on S descends exactly to the *normalized* Hamiltonian H_α on $Gr(r, n)$.

Assume now that $\alpha_{j_1} \neq \alpha_{j_2}$ for $j_1 \neq j_2$. This implies that H_α is a Morse function on $Gr(r, n)$. From the definition of \mathfrak{U} one sees that H_α also satisfies the conditions (B) and (C) of Definition 2.1.6. Thus H_α is a slow Morse Hamiltonian.

The considerations above allow us to compute easily the critical values of H_α . At a critical point z_I corresponding to a subset $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$ the critical value of H_α is equal to

$$H_\alpha(z_I) = - \sum_{j \in I} \alpha_j.$$

Hence

$$\mathcal{A}_\alpha([z_I, f_d]) = \sum_{j \in I} \alpha_j - d, \quad (19)$$

where $[z_I, f_d] \in \mathcal{P}(H_\alpha)$ is formed by a constant path z_I and a two-dimensional sphere f_d attached to z_I and representing the class $d \in \mathbf{Z} \cong \Pi$.

The critical points of H_α are the fixed points of the action $\{g_{e^{2\pi i t}}\}$ and therefore they are in one-to-one correspondence with the invariant complex r -dimensional subspaces of the matrix $\text{diag}(e^{2\pi i \alpha_1 t}, \dots, e^{2\pi i \alpha_n t})$ or, equivalently, with the subsets $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$. Thus the total number of the critical points of H_α is equal to the total sum of the Betti numbers of $Gr(r, n)$ and therefore H_α is a *perfect* Morse function on $Gr(r, n)$.

Let us now consider the isomorphism between the Morse homology of H_α and the singular homology of $Gr(r, n)$ and let us show that a critical point z_I corresponds to the homology class σ_I . Assume without loss of generality that the flag $\{0\} = F_0 \subset \dots \subset F_n = \mathbf{C}^n$ used to define the homology classes σ_I is the standard flag spanned by the basic vectors $\mathbf{e}_1, \dots, \mathbf{e}_r$ associated with the coordinates w_1, \dots, w_l on \mathbf{C}^n :

$$F_k = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_k),$$

$k = 1, \dots, n$. Given a subset $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$ consider the r -dimensional subspace $V_I \subset \mathbf{C}^n$ generated by the r basic vectors $\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_r}$ and the $(n-r)$ -dimensional subspace V_I^\perp spanned by the rest of the basic vectors \mathbf{e}_j . A neighborhood of V_I in $Gr(r, n)$ can be identified with the space $\mathbf{C}^{r(n-r)}$ of r -tuples $(\mathbf{v}_1, \dots, \mathbf{v}_r)$, $\mathbf{v}_j \in V_I^\perp$, $j = 1, \dots, r$: namely, to a tuple $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_r)$ one associates the point $x_{\mathbf{v}} \in Gr(r, n)$ corresponding to the r -dimensional subspace of \mathbf{C}^n spanned by $(\mathbf{e}_1 + \mathbf{v}_1, \dots, \mathbf{e}_r + \mathbf{v}_r)$. The

function f on a neighborhood of $x_0 = V_I$ in $Gr(r, n)$ defined by the formula

$$f(x_{\mathbf{v}}) = H''_{\alpha}(\mathbf{e}_1 + \mathbf{v}_1, \dots, \mathbf{e}_r + \mathbf{v}_r)$$

has a Morse critical point at x_0 and, under our choices of coordinates, the quadratic parts of f and H_{α} at $x_0 = V_I$ coincide. This can be seen if one applies the Gram-Schmidt procedure to turn the basis $(\mathbf{e}_1 + \mathbf{v}_1, \dots, \mathbf{e}_r + \mathbf{v}_r)$ of the subspace $x_{\mathbf{v}}$ into a unitary basis which can be viewed as an element of S . It allows us to check that the unstable manifold of H_{α} at $x_0 = V_I$, taken with respect to the natural Riemannian metric defined by the complex and the symplectic structures on the Grassmannian, coincides with W_I . Using this fact one gets that under the identification of the (integral) Morse homology of $-H_{\zeta^j}$ with the singular homology of $Gr(r, n)$ the homology class $z_I \in H_*(C_*(-H_{\zeta^j}))$ represented by z_I viewed as a critical point of $-H_{\zeta^j}$ gets identified with the class σ_{*I} , where $*I = \{n+1-i_1, \dots, n+1-i_r\} \subset \{1, \dots, n\}$ and σ_{*I} is the homology class defined by $*I$ which is in fact Poincaré-dual to σ_I .

Now let us finish the proof of the second inequality of the theorem. We prove it first for a *good* ζ , i.e. for a $\zeta \in \mathfrak{U}^l$ such that for each $\zeta^i = (\zeta_1^i, \dots, \zeta_n^i) \in \mathfrak{U}$, $1 \leq i \leq l$, one has $\zeta_{j_1}^i \neq \zeta_{j_2}^i$ for any $j_1 \neq j_2$. Such a ζ gives rise to slow Morse Hamiltonians $H_{\zeta} = (H_{\zeta^1}, \dots, H_{\zeta^l})$. Using our computations of the critical values of a slow Morse Hamiltonian H_{ζ^i} , $i = 1, \dots, l$, and the identification of its critical points with the homology classes of the Grassmannian as above one applies Corollary 2.2.2 to obtain the second inequality of the theorem in the case when ζ is good. Finally observe that the set of good ζ is open and dense in \mathfrak{U}^l . Therefore by continuity one gets the second inequality in the general case. ■

9 Proofs of the results concerning pseudo-holomorphic curves

We will briefly outline the proofs of the results concerning the moduli spaces of pseudo-holomorphic curves.

9.1 The proof of Proposition 6.3.1

The basic scheme of the proof is fairly standard for the Floer theory (see [10], [18] or [39]). We outline it for our particular case following M.Schwarz – the details can be found in [42], [18].

Preliminaries.

One defines $C_\gamma^\infty(\Sigma, M)$ as the space of smooth maps $F : \Sigma \rightarrow M$ such that over the end Σ_i , $i = 1, \dots, l$, one has

$$F \circ \Phi_i\left(\frac{s}{\sqrt{1-s^2}}, t\right) = \phi_i(s, t)$$

for some C^∞ -function ϕ_i on the *closed* cylinder $[0, 1] \times S^1$, where $\phi_i(1, \cdot) = \gamma_i$. The space $C_\gamma^\infty(\Sigma, M)$ can be completed to a separable infinite-dimensional Banach manifold $H_\gamma^{1,p}(\Sigma, M)$ modeled on the Sobolev space $H^{1,p}(\Sigma, \mathbf{R}^{2n})$ with $p > 2$.

Consider now the special case when $\Sigma = \mathbf{R} \times S^1$ is the standard cylinder. Let $h : S^1 \times M \rightarrow \mathbf{R}$ be a Hamiltonian function. We will say that h is *non-degenerate*, if all the contractible 1-periodic orbits of the Hamiltonian flow of h are non-degenerate (i.e. 1 is not a Floquet multiplier for any of the orbits). A generic Hamiltonian is non-degenerate. Now let J be an ω -compatible almost complex structure on M . Let γ', γ'' be contractible 1-periodic orbits of the Hamiltonian flow of h . Define the operator

$$\partial_{J,h} = \partial_s + J\partial_t - \nabla h$$

on $H_{\gamma', \gamma''}^{1,p}(\mathbf{R} \times S^1, M)$, where gradient is taken with respect to the Riemannian metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ on M . Consider the space $\mathcal{M}(\gamma', \gamma'') \subset H_{\gamma', \gamma''}^{1,p}(\mathbf{R} \times S^1, M)$ of solutions u of the equation

$$\partial_{J,h} u = 0.$$

We say that the pair (h, J) is *weakly regular*, if h is non-degenerate and the linearization of $\partial_{J,h}$ at any point $u \in \mathcal{M}(\gamma', \gamma'')$ is onto. A generic (with

respect to the C^∞ -topology) pair (h, J) is weakly regular [40]. Moreover, if h is a slow Morse Hamiltonian which is a Morse-Smale function with respect to the metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ then the pair (h, J) is weakly regular [40]. Given the Hamiltonians $H = (H_1, \dots, H_l)$ and the almost complex structure \tilde{J} as above we say that the pair (H, \tilde{J}) is *weakly regular* if for each $i = 1, \dots, l$ the pair $(H_i, J(\tilde{J}))$ is weakly regular.

Smoothness of $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$.

The operator $\bar{\partial}_{\tilde{J}}$ defined on $C_\gamma^\infty(\Sigma, M)$ can be extended to $H_\gamma^{1,p}(\Sigma, M)$. Consider the solutions $u \in H_\gamma^{1,p}(\Sigma, M)$ of the equation $\bar{\partial}_{\tilde{J}}u = 0$. If (H, \tilde{J}) is weakly regular then using local elliptic regularity and Sobolev embedding theorems one shows that any such solution has to lie inside $C_\gamma^\infty(\Sigma, M)$.

Let us view elements of $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ as maps $\Sigma \rightarrow M$. Then $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ can be viewed as the zero level set of the operator $\bar{\partial}_{\tilde{J}}$ defined on $H_\gamma^{1,p}(\Sigma, M)$.

Using the Fredholm theory one shows that if (H, \tilde{J}) is weakly regular then the linearization of the operator $\bar{\partial}_{\tilde{J}}$ on $H_\gamma^{1,p}(\Sigma, M)$ at any $u \in \mathcal{M}(\hat{\gamma}, H, \tilde{J})$ is Fredholm for any $p \geq 2$. For $p = 2$ the proof of this fact is fairly easy (see e.g. [10], [42]). For $p > 2$ the result is deduced in [42] from the case $p = 2$ by means of a local L^p -estimate for the Cauchy-Riemann operator (see e.g. [42], [39]).

To compute the index of the elliptic operator $\bar{\partial}_{\tilde{J}}$ one uses the index additivity with respect to gluing of trivial bundles over different surfaces along with the operators $\bar{\partial}_{\tilde{J}}$ on those bundles. Namely, one computes directly the index for the operator $\bar{\partial}_{\tilde{J}}$ on the bundle over a plane (an open disc with the cylindrical end). Then one caps off Σ with the discs (gluing together the bundles and the operators over the cylindrical ends) and reduces the problem to the case of a $\bar{\partial}$ -operator over a closed Riemann surface, where the answer is given by the Riemann-Roch formula. Finally one gets that the index of the operator $\bar{\partial}_{\tilde{J}}$ on the bundle over Σ is equal to $2n - \mu(\hat{\gamma})$.

Thus the implicit function theorem for Banach manifolds implies that if the linearization of $\bar{\partial}_{\tilde{J}}$ is onto at any $u \in \mathcal{M}(\hat{\gamma}, H, \tilde{J})$ then $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ is a smooth manifold of dimension $2n - \mu(\hat{\gamma})$. In such a case we say that a (weakly regular) pair (H, \tilde{J}) is *regular*.

Transversality.

A transversality result based on an infinite-dimensional version of the Sard theorem shows that regular pairs form a residual set in the set of all weakly regular pairs. Here is the first place where we have to do something to put the discussion in the framework of [42] (recall that our class of almost complex structures is larger than in [42]) – all the previous statements were

literally covered by the results in [42]. We can assume without loss of generality that the symplectic form $\Omega_{\tilde{\omega}, \tau}$ on $\Sigma \times M$, with which our almost complex structures \tilde{J} are compatible, coincides with the split form $\pi^* \Omega \oplus \omega$ outside of a compact set contained in the complement $\Sigma^0 = \Sigma \setminus \bigcup_{i=1}^l \Sigma_i$. In [42] the strategy was to perturb the extension of the Hamiltonians H_1, \dots, H_l over $\pi^{-1}(\Sigma^0)$ that had been defined by means of the cut-off function β in order to perturb the term corresponding to the gradients of the Hamiltonians in the $\bar{\partial}$ -operator. Then one considered in [42] variations of the almost complex structures J_x on the fibers $\pi^{-1}(x)$, $x \in \Sigma^0$, to get the necessary result. In our case, due to our definition of $\mathcal{T}(H)$, the perturbation of the Hamiltonian term in the $\bar{\partial}$ -operator can be viewed as the result of a perturbation of the almost complex structure $\tilde{J} \in \mathcal{T}(H)$. Then one follows the proof of Theorem 4.2.20 in [42] and obtains the following result: given a weakly regular pair (H, \tilde{J}) one can always find an open set $K \subset \Sigma$ with compact closure so that for a generic and arbitrarily C^∞ -small perturbation $\tilde{J}_1 \in \mathcal{T}(H)$ of \tilde{J} over K the pair (H, \tilde{J}_1) is regular. Observe also that a sufficiently small perturbation $\tilde{J}_1 \in \mathcal{T}(H)$ of \tilde{J} remains compatible with the symplectic form $\Omega_{\tilde{\omega}, \tau}$.

Compactness.

As before let $\pi : \Sigma \times M \rightarrow \Sigma$, $pr_M : T_*(\Sigma \times M) \rightarrow T_*M$ be the natural projections and let J_x be the restriction of \tilde{J} on $\pi^{-1}(x)$. Let the norm $\|\cdot\|_x$ on the tangent bundle of the fiber $\pi^{-1}(x)$ be defined by the metric $\omega(\cdot, J_x \cdot)$.

Given a map $u : \Sigma \rightarrow \Sigma \times M$ from $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ and a conformal chart $f : U \rightarrow \Sigma$, $U \in \mathbf{C}$, with coordinates s, t set

$$u_s = pr_M \circ \frac{\partial(u \circ f)}{\partial s}, u_t = pr_M \circ \frac{\partial(u \circ f)}{\partial t}$$

and define the 2-form $\|du\|\Omega$ on $f(U)$, where Ω is the volume form on Σ , as

$$\|du\|\Omega(x) = f^* \{ (\|u_s(x)\|_x + \|u_t(x)\|_x) ds \wedge dt \}.$$

One can check that the form $\|du\|\Omega$ defined in such a way is a correctly defined 2-form on the whole Σ . Define the *energy* of u as

$$E(u) = \int_{\Sigma} \|du\|\Omega.$$

Compactness of the space $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ can be shown by the following standard argument (see [18], [19], [42]). First of all one checks that there is an *a priori* uniform bound on the energies of all $u \in \mathcal{M}(\hat{\gamma}, H, \tilde{J})$. (see [42],

also see [2], Lemma 5.2). Now there are only two phenomena that might obstruct the compactness of $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$: bubbling-off of \tilde{J} -holomorphic spheres or convergence to a “broken trajectory” (see e.g. [10], [18], [19], [39] or [42]). The latter obstacle does not occur in our case since we consider only zero-dimensional moduli spaces $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ (otherwise one has to use Floer’s gluing techniques – see [10], also see [41]). As far as the bubbling-off is concerned one quickly observes that because of our definition of $\mathcal{T}(H) \supset \tilde{J}$ the projection of such a bubble on Σ has to be holomorphic and thus the maximum principle dictates that the bubble has to lie inside a fiber of $\Sigma \times M \rightarrow \Sigma$. Then one uses the fact that (M, ω) is strongly semi-positive and therefore (see [18], [45]) a generic 2-parametric family of ω -compatible almost complex structures on (M, ω) does not contain an almost complex structure J which admits a J -holomorphic sphere of negative Chern number. Such a 2-parametric family arises from \tilde{J} since the restriction of \tilde{J} may vary with the base point in the two-dimensional surface Σ . Also because of the transversality reasons one can always assume without loss of generality that the 1-periodic trajectories of the Hamiltonian flows of H_1, \dots, H_l do not intersect the $J(\tilde{J})$ -holomorphic spheres with Chern number 1 in M . Then one proceeds as in [18] (cf. [43]) and shows that for such a \tilde{J} in the absence of pseudo-holomorphic curves with negative Chern numbers the bubbling-off in the fibers cannot occur, which shows the compactness of $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$.

Orientation.

The orientation can be obtained by the methods from [11].

This finishes the proof of Proposition 6.3.1. ■

9.2 The proof of Theorem 6.3.2

Pick a connection $\nabla \in \mathcal{L}_{\{c\}}^{fit}([\varphi_H])$, the corresponding 2-form $\tilde{\omega}_\nabla = \tilde{\omega}$, such that $\tau_0 < \text{size}(\tilde{\omega})$, and a weak coupling deformation $\{\Omega_{\tilde{\omega}, \tau'}\}$, $0 < \tau' < \tau$. Observe that for τ' close to zero almost complex structures from $\mathcal{T}_0(H)$ are compatible with $\Omega_{\tilde{\omega}, \tau'}$.

Now the transversality and compactness results from the proof of Proposition 6.3.1 in Section 9.1 also hold for 1-parametric families of almost complex structures from $\mathcal{T}(H)$. (The compactness result relies on the fact that a generic 2-parametric family of ω -compatible almost complex structures on (M, ω) does not contain an almost complex structure J which admits a

J -holomorphic sphere of negative Chern number). This allows us to choose some small $\epsilon > 0$ and a family $\{\tilde{J}_{\tau'}\}$, $\epsilon \leq \tau' \leq \tau$, so that:

- Each $\tilde{J}_{\tau'}$ belongs to $\mathcal{T}_J(H)$ and is compatible with $\Omega_{\tilde{\omega}, \tau'}$.
- For all τ' sufficiently close to ϵ one has $\tilde{J}_{\tau'} \in \mathcal{T}_0(H)$.
- For all τ' sufficiently close to ϵ and to τ the pair $(H, \tilde{J}_{\tau'})$ is regular.
- For any $\hat{\gamma}$ with $\mu(\hat{\gamma}) = 2n$ the union

$$\tilde{\mathcal{M}}(\hat{\gamma}, H, \tilde{J}_{\tau'}) = \bigcup_{\epsilon \leq \tau' \leq \tau} \mathcal{M}(\hat{\gamma}, H, \tilde{J}_{\tau'})$$

forms a compact (oriented) 1-dimensional manifold with boundary (where $\mathcal{M}(\hat{\gamma}, H, \tilde{J}_{\epsilon})$ and $\mathcal{M}(\hat{\gamma}, H, \tilde{J}_{\tau})$ are among the connected components of the boundary).

- For any $\hat{\gamma}$ with $\mu(\hat{\gamma}) = 2n + 1$ the union

$$\tilde{\mathcal{M}}(\hat{\gamma}, H, \tilde{J}_{\tau'}) = \bigcup_{\epsilon \leq \tau' \leq \tau} \mathcal{M}(\hat{\gamma}, H, \tilde{J}_{\tau'})$$

is a compact (oriented) 0-dimensional manifold.

Now for τ' close to ϵ , when $\tilde{J}_{\tau'} \in \mathcal{T}_0(H)$, the theorem is literally the result from [28]. To extend it to other τ' one uses the standard homotopy argument from the Floer theory as in [42] (Ch. 5.2). Namely, one constructs a chain endomorphism of $CF_*(H_1, J) \otimes \dots \otimes CF_*(H_l, J)$ by counting points in zero-dimensional spaces $\tilde{\mathcal{M}}(\hat{\gamma}, H, \tilde{J}_{\tau'})$. This endomorphism maps the chain $\theta_{\Sigma, H, \tilde{J}_{\epsilon}}$ into the chain $\theta_{\Sigma, H, \tilde{J}_{\tau}}$. Then one constructs a chain homotopy of the endomorphism to the identity by means of the 1-dimensional spaces $\tilde{\mathcal{M}}(\hat{\gamma}, H, \tilde{J}_{\tau'})$ thus showing that the homology classes of the cycles realized by $\theta_{\Sigma, H, \tilde{J}_{\epsilon}}$ and $\theta_{\Sigma, H, \tilde{J}_{\tau}}$ coincide. ■

9.3 The proof of Proposition 6.3.3

We use the same kind argument that was used in [28] in the proof of the statement that \mathcal{QF}_h is an isomorphism between quantum and Floer (co)homology. For a slow Morse Hamiltonian h one can choose a *time-independent* almost complex structure J on M so that the pair (h, J) is regular.

To define the map \mathcal{QF}_h (see [28], the proof of Theorem 4.1) one first identifies the quantum homology with the homology of the Morse complex

of h with coefficients in Λ_ω . Since the Morse function h is slow its Morse complex can be canonically identified with its Floer complex $CF_*(h, J)$. Then one constructs \mathcal{QF}_h as the map in homology induced by a chain endomorphism of that complex. (For simplicity we denote by \mathcal{QF}_h the induced maps both in homology and cohomology). Thus we need to prove that this chain endomorphism induces the identity map in the homology of the Morse complex. We will show that it is actually identity already on the level of chains.

Indeed, recall how \mathcal{QF}_h is defined as a chain endomorphism. One considers the moduli space $\mathcal{M}_{\mathbf{C}}(y, h, J)$ of “spiked disks”, i.e. the space defined as in Definition 4.0.7 with $l = 1$, $\Sigma = \mathbf{C}$, where y is a critical point of h . Then one picks critical points x, y of h in such a way that the space of curves from $\mathcal{M}_{\mathbf{C}}(y, h, J)$ with $u(0)$ lying on the unstable manifold of a critical point x is zero-dimensional. Counting such curves in various spherical non-torsion homology classes A , such that $\text{ind } z - \text{ind } x = 2c_1(A)$, one defines the coefficients $\langle x, y, A \rangle$ that determine the map \mathcal{QF}_h .

Consider the deformation τh , $0 \leq \tau \leq 1$. The number $\langle x, y, A \rangle$, defined by means of the Hamiltonian τh , does not change as τ goes from one to zero. But when τ is zero the number $\langle x, y, A \rangle$ counts J -holomorphic curves in the homology class A . Therefore, since (M, ω) is strongly semi-positive, $c_1(A)$ has to be non-negative for any A such that $\langle x, y, A \rangle \neq 0$.

Now, if one uses a time-independent J then, since h is also time-independent, any curve from $\mathcal{M}_{\mathbf{C}}(y, h, J)$ with $u(0)$ lying on the unstable manifold of a critical point x can be “rotated” by changing the time parameter and in this way one gets a one-dimensional family of solutions. This leads to a contradiction with the zero dimension of $\mathcal{M}_{\mathbf{C}}(y, h, J)$, unless the original curve which we rotated was time-independent and hence was a part of the anti-gradient trajectory of h going from x to y . But since $c_1(A)$ is always non-negative, $\text{ind } z - \text{ind } x \geq 0$ and therefore we must have $x = y$, $A = 0$, which shows that \mathcal{QF}_h is identity already on the level of chains. ■

10 Proof of Proposition 3.5.3

Given a tuple $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_l)$ of conjugacy classes in G and a tuple $\phi = (\phi_1, \dots, \phi_l)$ of elements from G we will write $\phi \in \mathcal{C}$ if $\phi_i \in \mathcal{C}_i$, $i = 1, \dots, l$. Denote by Δ the set of all tuples $f = (f_1, \dots, f_l)$ of elements from G such that $f_1 \cdot \dots \cdot f_l = Id$.

Consider systems of paths $a = (a_1, \dots, a_l) \in \mathcal{G}(\mathcal{C})$ containing $l - 1$ constant paths a_i , $i = 2, \dots, l$, identically equal to $\phi_i \in \mathcal{C}_i$. Set $\phi_1 = a_1(1) \in \mathcal{C}_1$. Then

$$length(a) = length(a_1) \geq \rho(Id, a_1(1) \cdot a_1^{-1}(0)) = \rho(Id, \prod_{i=1}^l \phi_i).$$

Thus we get that

$$\Upsilon_l(\mathcal{C}) = \inf_{\phi \in \mathcal{C}} \rho(Id, \prod_{i=1}^l \phi_i) \geq \inf_{a \in \mathcal{G}(\mathcal{C})} length(a).$$

Let us prove the opposite inequality. Because of the bi-invariance of the pseudo-metric and the elementary property $\Upsilon_l(\mathcal{C}) = \Upsilon_l(\mathcal{C}^{-1})$ it suffices to show that

$$\inf_{f \in \Delta, \psi \in \mathcal{C}} \sum_{i=1}^l \rho(Id, f_i \psi_i^{-1}) \geq \inf_{\phi \in \mathcal{C}} \rho(Id, \prod_{i=1}^l \phi_i^{-1}). \quad (20)$$

Using the triangular inequality one gets:

$$\inf_{f \in \Delta, \psi^{-1} \in \mathcal{C}} \sum_{i=1}^l \rho(Id, f_i \psi_i^{-1}) \geq \inf_{f \in \Delta, \psi \in \mathcal{C}} \rho(Id, \prod_{i=1}^l f_i \psi_i^{-1})$$

Now set $F_i = \prod_{j=1}^i f_j$, $i = 1, \dots, l$, use the identity $F_l = \prod_{i=1}^l f_i = Id$ and observe that

$$\prod_{i=1}^l f_i \psi_i^{-1} = \left(\prod_{i=1}^l F_i \psi_i^{-1} F_i^{-1} \right) \cdot F_l = \phi_1^{-1} \cdot \dots \cdot \phi_l^{-1}$$

for some $\phi = (\phi_1, \dots, \phi_l) \in \mathcal{C}$. This implies (20) and the proposition is proven. ■

11 Proof of Theorem 3.4.1

11.1 The case of a cylinder: $l = 2$

In the case when Σ is a cylinder the proof basically imitates the similar proofs from [30], [32].

11.1.1 Coarse length

Let $\gamma : [0, 1] \rightarrow G$ be a smooth path in G . Let $\|\cdot\|$ be the Finsler norm on the tangent bundle of G defining our bi-invariant Finsler pseudo-metric on the group. Besides the usual length which is defined as

$$\text{length}(\gamma) = \int_0^1 \left\| \frac{d\gamma}{ds} \right\| ds$$

one can define a quantity *coarse-length*(γ) as

$$\text{coarse-length}(\gamma) = \max_{s \in [0, 1]} \|d\gamma/ds\|.$$

Obviously one always has

$$\text{length}(\gamma) \leq \text{coarse-length}(\gamma).$$

Observe that *coarse-length*(γ), unlike *length*(γ), essentially depends on the parameterization of the path $\gamma : [0, 1] \rightarrow G$. However, by reparameterizing γ one can always make its coarse length equal to its length (see e.g. [33]).

11.1.2 Preliminaries

Without loss of generality we can assume that Σ is a standard cylinder $[0, 1] \times S^1$ with the coordinates (s, t) , $0 \leq s \leq 1$, $0 \leq t \leq 1 \pmod{1}$, and equipped with the area form $\Omega = ds \wedge dt$, so that $\int_{\Sigma} \Omega = 1$.

Set $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$. Recall from Definition 3.3.1 that $\mathcal{L}(\mathcal{C})$ is the set of connections on $P = \Sigma \times F \rightarrow \Sigma$ which are flat near the boundary and whose holonomies over the closed paths $\{t \rightarrow 0 \times t\}_{0 \leq t \leq 1}$ and $\{t \rightarrow 1 \times t\}_{0 \leq t \leq 1}$ belong, respectively, to the conjugacy classes \mathcal{C}_1^{-1} and \mathcal{C}_2 . We denote by $\mathcal{G}'(\mathcal{C})$ the set of all smooth paths $a : [0, 1] \rightarrow G$ such that $a(0) \in \mathcal{C}_1^{-1}$, $a(1) \in \mathcal{C}_2$.

By $\mathcal{G}'_{\{a\}}(\mathcal{C})$ we denote the connected component of $\mathcal{G}'(\mathcal{C})$ corresponding to the homotopy class $\{a\}$ of paths connecting \mathcal{C}_1^{-1} and \mathcal{C}_2 .

In our case, when $l = 2$, the sets $\mathcal{G}(\mathcal{C})$ and $\mathcal{G}'(\mathcal{C})$ are closely related. Indeed, an element of $\mathcal{G}(\mathcal{C}_1, \mathcal{C}_2)$ is a pair of paths $a_1, a_2 : [0, 1] \rightarrow G$ such that $a_1^{-1}(0) = a_2(0)$, $a_1(1) = \phi_1 \in \mathcal{C}_1$, $a_2(1) = \phi_2 \in \mathcal{C}_2$. Joining the curves a_1^{-1} and a_2 in G at their common point $a_1^{-1}(0) = a_2(0)$ one obtains a curve that connects ϕ_1^{-1} and ϕ_2 . Conversely, given a path $c : [0, 1] \rightarrow G$ connecting ϕ_1^{-1} and ϕ_2 , one can view it as a union of two curves that join each other at their common endpoint. One of these curves can be taken for a_2 and the group inverse of the other one for a_1 . Appropriately parameterized these two new curves form a system of paths belonging to $\mathcal{G}(\mathcal{C}_1, \mathcal{C}_2)$. One easily sees that there is a one-to-one correspondence between homotopy classes of elements of $\mathcal{G}(\mathcal{C})$ and $\mathcal{G}'(\mathcal{C})$.

Given a connection $\nabla \in \mathcal{L}(\mathcal{C})$ and a trivialization \tilde{P} of $P \rightarrow \Sigma$, one can associate to it a path $a_{\nabla, \tilde{P}}$ belonging to $\mathcal{G}'(\mathcal{C})$. Namely let $\varphi_1^{-1} \in \mathcal{C}_1^{-1}$ and $\varphi_2 \in \mathcal{C}_2$ be the holonomies of ∇ (with respect to the fixed trivialization \tilde{P}) along, respectively, the closed paths $\{0 \times t\}_{0 \leq t \leq 1}$ and $\{1 \times t\}_{0 \leq t \leq 1}$. Define a path $a_{\nabla, \tilde{P}} : [0, 1] \rightarrow G$ by taking the holonomy of ∇ along the closed path $\{t \rightarrow s \times t\}_{0 \leq t \leq 1}$ as $a_{\nabla, \tilde{P}}(s)$. The homotopy class of $a_{\nabla, \tilde{P}}$ depends only on $[\nabla]$ and will be denoted by $\{\nabla\}$. The corresponding connected component of $\mathcal{G}'(\mathcal{C})$ will be denoted by $\mathcal{G}'_{\{\nabla\}}(\mathcal{C})$. We will show in the Section 11.1.4 that any homotopy class of paths from $\mathcal{G}'(\mathcal{C})$ can be represented as $\{\nabla\}$ for some $[\nabla]$.

Now the theorem (for the case of a cylinder) becomes an immediate corollary of the following lemma.

Lemma 11.1.2.1

$$\inf_{\nabla \in \mathcal{L}_{[\nabla]}(\mathcal{C})} \|L^\nabla\| = \inf_{a \in \mathcal{G}'_{\{\nabla\}}(\mathcal{C})} \text{coarse-length}(a) \quad (21)$$

11.1.3 Proof of \geq in (21)

Let us take a connection $\nabla \in \mathcal{L}(\mathcal{C})$. Let us choose a global trivialization \tilde{P} of $P \rightarrow \Sigma$ so that the holonomy of ∇ along any interval inside $[0, 1] \times 0$ is identity. Given the trivialization, let φ_1^{-1} and φ_2 be the holonomies of ∇ along the closed paths $\{t \rightarrow 0 \times t\}_{0 \leq t \leq 1}$ and $\{t \rightarrow 1 \times t\}_{0 \leq t \leq 1}$ respectively. Then,

as it was described above, ∇ defines a path $a_{\nabla, \tilde{P}} : [0, 1] \rightarrow G$ connecting φ_1^{-1} and φ_2 and belonging to $\mathcal{G}'_{\{\nabla\}}(\mathcal{C})$.

In order to prove (21) it is enough to prove

$$\|L^\nabla\| \geq \max_s \left\| \frac{da_{\nabla, \tilde{P}}}{ds}(s) \right\|. \quad (22)$$

To prove (22) let us denote by $\partial/\partial s$ and $\partial/\partial t$ the standard vector fields on $\Sigma = [0, 1] \times S^1$ and by X and Y their horizontal lifts. Let X^s and Y^t be the vertical components of the flows of X and Y respectively. Set $\phi_{s,t} = Y^t X^s$ (this map is defined on a domain which depends on s and t). Consider a family $v_{s,t}$ of vector fields on the fiber:

$$v_{s,t} = \frac{\partial \phi_{s,t}}{\partial s} = Y_*^t X.$$

Then, since the holonomy of ∇ along any interval inside $[0, 1] \times 0$ is identity, one has

$$v_{s,1}(s, 1) = \frac{da_{\nabla, \tilde{P}}}{ds}(s)$$

and

$$v_{s,0}(s, 0) = 0.$$

On the other hand,

$$\frac{\partial v_{s,t}}{\partial t} = Y_*^t [X, Y]^{vert},$$

where $[X, Y]^{vert}$ is the vertical component of $[X, Y]$. Thus

$$\frac{da_{\nabla, \tilde{P}}}{ds}(s) = v_{s,0}(s, 0) + \int_0^1 \frac{\partial v_{s,1}}{\partial t}(s, 0) dt = \int_0^1 Y_*^t [X, Y]^{vert}(s, 0) dt,$$

and hence

$$\int_0^1 \|Y_*^t [X, Y]^{vert}(s, 0)\| dt \geq \left\| \frac{da_{\nabla, \tilde{P}}}{ds}(s) \right\|. \quad (23)$$

The vector field $[X, Y]^{vert}$, viewed by means of the trivialization of the bundle as an element of \mathfrak{g} , is equal (up to sign) to the curvature $L^\nabla(\partial/\partial s, \partial/\partial t)$. Since the Finsler norm on G is *bi-invariant* we get that

$$\|Y_*^t [X, Y]^{vert}(s, 0)\| \leq \|L^\nabla\|. \quad (24)$$

Combining (23) and (24) we get (22) and thus prove (21). ■

11.1.4 Proof of \leq in (21).

Let us take a path $a : [0, 1] \rightarrow G$ belonging to $\mathcal{G}'(\mathcal{C})$. Reparameterize a so that it becomes constant near 0 and 1. Let $K = [0, 1] \times [0, 1]$. The trivial bundle $P \rightarrow \Sigma$ can be constructed by taking the bundle $K \times F \rightarrow K$ and identifying $(s, 0) \times a(s)y$ with $(s, 1) \times y$ for all $s \in [0, 1]$, $y \in G$. (Here $a(s)y$ denotes the action of the element $a(s) \in G$ on $y \in F$).

We are going to define a connection ∇ on $K \times F \rightarrow K$ by defining the corresponding field of horizontal planes. Take a monotone cut-off function $\psi(t)$ on the segment $[0, 1]$ such that $\psi(t) = 1$ when t is near 0, and $\psi(t) = 0$ when t is near 1.

The derivative $\frac{da}{ds}(s)$ is a vector tangent to G at the point $a(s) \in G$. It can be identified with an element $a'(s) \in \mathfrak{g}$ corresponding to the right-invariant vector field on G containing $\frac{da}{ds}(s)$. On the other hand, if one considers the action of the elements $a(s)$ on F then the vector $\frac{da}{ds}(s)$ can be viewed as a vector field on F whose value at a point y will be denoted by $A(s, y)$.

Our horizontal plane will be generated by two horizontal vector fields X and Y which are the horizontal lifts of $\partial/\partial s$ and $\partial/\partial t$ respectively. The vector fields are defined as follows: at a point $(s, t) \times y \in K \times F$ our horizontal plane will be generated by the vectors

$$X_{(s,t) \times y} = \langle 1, 0, \psi(t)A(s, y) \rangle$$

and

$$Y_{(s,t) \times y} = \langle 0, 1, 0 \rangle.$$

One easily checks that the plane field spanned by X and Y defines a G -connection ∇ on $P \rightarrow \Sigma$ which is flat near the boundary and whose holonomies over the ends of Σ belong to the conjugacy classes \mathcal{C}_1 and \mathcal{C}_2 . Therefore $\nabla \in \mathcal{L}(\mathcal{C})$ and clearly $[a] = \{\nabla\}$. This shows in particular that any homotopy class of paths from $\mathcal{G}'(\mathcal{C})$ can be realized as $\{\nabla\}$ for some $[\nabla]$.

The commutator of X and Y is a vertical vector field which at a point $(s, t) \times y \in K \times F$ looks as follows:

$$[X, Y]_{(s,t) \times y} = \langle 0, 0, \psi'(t)A(s, y) \rangle.$$

The curvature $L^\nabla(\partial/\partial s, \partial/\partial t)$ at the point (s, t) is the element $a'(s) \in \mathfrak{g}$. Therefore, since the Finsler norm on G is *bi-invariant* one has

$$\|L^\nabla(s, t)\| = |\psi'(t)| \cdot \|a'(s)\| = |\psi'(t)| \cdot \left\| \frac{da}{ds}(s) \right\|.$$

Hence

$$\|L^\nabla\| = \max_{(s,t)} \|L^\nabla(s,t)\| \leq \max_t |\psi'(t)| \cdot \max_s \left\| \frac{da}{ds}(s) \right\|.$$

Now fix a small positive number ε . We can choose the function ψ above so that $|\psi'(t)| \leq 1 + \varepsilon$ for all t . Therefore we get

$$\|L^\nabla\| \leq (1 + \varepsilon) \text{ coarse-length}(a).$$

Since ε was taken arbitrarily we get the desired inequalities. This finishes the proof of Lemma 11.1.2.1 and the proof of Theorem 3.4.1 in the case when Σ is a cylinder. ■

Remark 11.1.1 Suppose that one drops the normalization condition for the area of the cylinder Σ to be 1. Then one can repeat the proofs above for the cylinder $[0, A] \times S^1$ of area A taking the rescaling into account. As a result one would get the inequalities:

$$A \cdot \inf_{\nabla \in \mathcal{L}(\mathcal{C})} \|L^\nabla\| \geq \inf_{a \in \mathcal{G}'(\mathcal{C})} \text{length}(a).$$

11.2 The case $l \neq 2$

11.2.1 The case of a disc: $l = 1$

Let Σ be a disc. Present it as a cylinder Cyl one of whose boundary components is capped with a disc D . Considering trivial flat connections on $P \rightarrow \Sigma$ defined over D , gluing them with connections defined over Cyl to get a connection over the whole Σ and using Lemma 11.1.2.1 and Remark 11.1.1 one easily proves the theorem for the case $l = 1$. ■

11.2.2 The case $l > 1$

Let us cut Σ into cylinders Cyl_1, \dots, Cyl_l (with piecewise smooth boundaries) in the following way. Each boundary component of Σ will be a boundary component of exactly one of these cylinders. If $l = 1$ we view the disc Σ as a cylinder with one boundary component capped with a disc. Otherwise, if $l > 1$, we make the cuts in such a way that the boundaries of the cylinders are not smooth only at some two common points $p, \bar{p} \in \Sigma$. The union of

all such boundaries passing through p and \bar{p} consists of l paths $\theta_1, \dots, \theta_l$ coming out in the counterclockwise order from p and connecting it with \bar{p} . The closed paths $c_1 = \theta_1 \circ \theta_1^{-1}, c_2 = \theta_2 \circ \theta_1^{-1}, \dots, c_l = \theta_l \circ \theta_{l-1}^{-1} : S^1 \rightarrow \Sigma$ are, respectively, the boundary components of the cylinders Cyl_1, \dots, Cyl_l (the other boundary components of these cylinders are, respectively, T_1, \dots, T_l). Without loss of generality we can assume that the areas of the cylinders Cyl_1, \dots, Cyl_l are all equal to $\frac{1}{l}$.

Consider the standard cylinder $[0, 1] \times S^1$ with the coordinates (s, t) , $0 \leq s \leq 1$, $0 \leq t \leq 1 \pmod{1}$, equipped with the area form $\frac{ds \wedge dt}{l}$ so that the total area is $1/l$.

For each $i = 1, \dots, l$ let us fix a point $y_i \in T_i$, and a map $\Phi_i : [0, 1] \times S^1 \rightarrow Cyl_i$ satisfying the following conditions:

- each $\Phi_i : [0, 1] \times S^1 \rightarrow Cyl_i$ extends the corresponding map $\Phi_i : [1 - \delta, 1] \times S^1 \rightarrow \Sigma$ from Section 3.1;
- $\Phi_i(0 \times t) = \Phi_{i+1}(0 \times \{1 - t\}) = c_i(t)$, $i = 1, \dots, l - 1$, $\Phi_l(0 \times t) = \Phi_1(0 \times \{1 - t\})$;
- $\Phi_i(1 \times 0) = y_i$, $\Phi_i(0 \times 0) = p$; $\Phi_i(0 \times 1/2) = \bar{p}$;
- the map Φ_i is an *area-preserving* diffeomorphism over the pre-image of $Cyl_i \setminus \{p, \bar{p}\}$ and a homeomorphism over the closed cylinder.

To a connection ∇ on $P = \Sigma \times G \rightarrow F$ with a chosen trivialization \tilde{P} one can associate an l -system of paths $a_{\nabla, \tilde{P}}$ in G . Namely, suppose $\nabla \in \mathcal{L}(\mathcal{C})$. Set $a_i(s)$ to be the parallel transport of ∇ along the path $t \rightarrow \Phi_i(s \times t)$, $i = 1, \dots, l$. One can check that the paths $a_i(s)$, $0 \leq s \leq 1$, $i = 1, \dots, l$, together form an l -system of paths $a_{\nabla, \tilde{P}} \in \mathcal{G}(\mathcal{C})$.

As we mentioned in Section 3.5 the space $\mathcal{G}(\mathcal{C})$ might have more than one connected component. One can check that if instead of \tilde{P} we take another trivialization then the new system of paths from $\mathcal{G}(\mathcal{C})$ is homotopic to the old one. Thus the homotopy class of $a_{\nabla, \tilde{P}}$ depends only on the homotopy class $[\nabla]$ and will be denoted by $[[\nabla]] = [a]([\nabla])$.

Gluing connections on $P \rightarrow G$ defined over the cylinders Cyl_1, \dots, Cyl_l into a connection defined over the whole Σ one can show that any homotopy class $[a]$ can be represented as $[[\nabla]] = [a]([\nabla])$ for some $[\nabla]$. Then using the same gluing technique together with Lemma 11.1.2.1 and Remark 11.1.1 one easily proves the theorem for the case $l > 1$.

This finishes the proof of Theorem 3.4.1. ■

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